Class Notes on Renormalization
Useful references include Peskin and Schroeder, Ryder and Cheng and Li.

Jack Gunion
U.C. Davis

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Introductory Ideas

Renormalization is not unique to QFT. For example, recall the difference between m and m^* (the effective mass) for an electron in a solid. $m \to m^*$ as a result of the electron's interaction with the lattice ions as it moves through the lattice.

However, there is an important difference relative to QFT. This is the fact that both m and m^* are, in principle, measurable. In particular, the bare mass m (i.e. outside the lattice) can be measured in the usual way. In QFT, we will find that:

1. The bare mass cannot be directly measured — only the renormalized mass is measurable.

   There is no way to switch off the interactions. The electron is always exposed to the QED interactions through virtual processes involving virtual photons and virtual electron-positron loops.

2. $m - m^*$ is naively infinite until some cutoff due to the physics of very large momentum (small distances) is brought into the picture.

If we eventually acquire a full understanding of physics at all momentum scales, we will be able to *compute* the bare mass in terms of the measured renormalized mass m^*.

Thus, in QFT, the process of renormalization consists of:

1. **Imagining that we will ultimately understand the high scale physics and in particular taking seriously the idea that there will be some way of cutting off the divergent integrals at high momentum.**

2. **Understanding that the exact form of this cutoff is not important when dealing with observations at energies well below the high scale of the cutoff physics, so long as one can rephrase all low energy predictions in terms of a small number of quantities measured at low energy.**

   Theories with this property are called *renormalizable* theories.

3. **Rewriting all predictions in terms of a few measured quantities (e.g. in QED the mass and charge of the electron as measured in some particular experiment). Since the measured quantities are finite, all our predictions will be finite and well defined in terms of these few inputs.**

Renormalization for $\lambda\phi^4$

By considering this relatively simple theory we will be able to understand essentially all the important basic procedures and ideas without the complexity of spinors and gauge fields.

The theory is defined by the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \tag{1}$$

with

$$\begin{aligned} \mathcal{L}_0 &= \frac{1}{2}[(\partial_\mu \phi_0)(\partial^\mu \phi_0) - \mu_0^2 \phi_0^2] \\ \mathcal{L}_I &= -\frac{\lambda_0}{4!}\phi_0^4 . \end{aligned} \tag{2}$$

Here, I have been careful to use 0 subscripts to denote the "bare" field, mass and coupling.

The bare propagator and four-point Feynman interaction are, respectively,

$$i\Delta_0(p) = \frac{i}{p^2 - \mu_0^2 + i\epsilon}, \quad -i\lambda_0 . \tag{3}$$

1PI Diagrams

An important concept is that of one-particle irreducible diagrams. 1PI means that you start with a single line and end with a single line and draw all diagrammatic structure in between which cannot be cut apart by cutting just a single line.

The complete set of insertions into a single line is then given by iterations of the 1PI diagram set with intermediate single line propagators.

$$\text{1 particle propagator} = \text{bare line} + \text{bare line}(-i\Sigma(p)\text{bare line} + \text{bl}(-i\Sigma(p))\text{bl}(-i\Sigma(p))\text{bl} + \ldots . \tag{4}$$

Here, $-i\Sigma(p)$ is the algebraic expression for the 1PI diagram set. This can be written algebraically as:

$$\begin{aligned} i\Delta(p) &= i\Delta_0(p) + i\Delta_0(p)[-i\Sigma(p)]i\Delta_0(p) + \ldots \\ &= \frac{i}{p^2 - \mu_0^2 + i\epsilon} + \frac{i}{p^2 - \mu_0^2 + i\epsilon}[-i\Sigma(p)]\frac{i}{p^2 - \mu_0^2 + i\epsilon} + \ldots \end{aligned}$$

$$
\begin{aligned}
&= \frac{i}{p^2 - \mu_0^2 + i\epsilon}\left[\frac{1}{1 + i\Sigma(p^2)\frac{i}{p^2-\mu_0^2+i\epsilon}}\right] \\
&= \frac{i}{p^2 - \mu_0^2 - \Sigma(p^2) + i\epsilon}\,,
\end{aligned} \tag{5}
$$

where, midway through, I used the fact that Lorentz invariance and analyticity imply the Σ can really only be a function of p^2.

Now, if we just simply write down an expression using Feynman rules of the lowest order contribution to $-i\Sigma(p^2)$ (coming from the bubble tadpole correction diagram), the expression is infinite (as we shall dwell on later). The ultimate high scale physics theory will make this expression finite. Effectively the divergence will be regulated by some high scale, but for the moment we can employ any regularization procedure that renders the diagram finite.

There are also divergent 1-loop corrections to the ϕ^4 interaction. All the relevant 1-loop diagrams appear below.

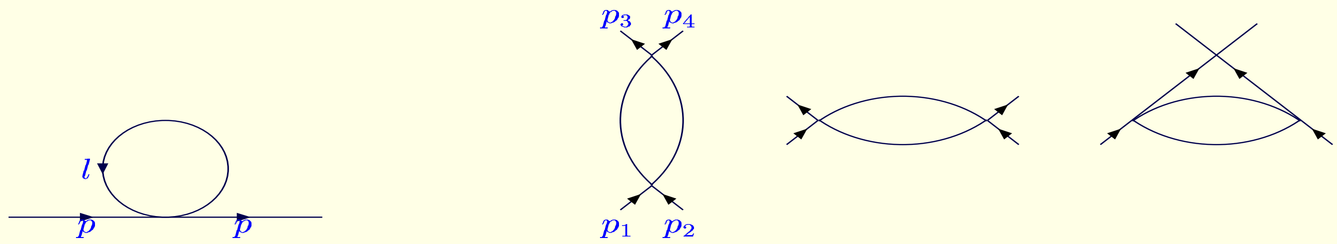


Figure 1: The one-loop corrections to the propagator and vertex.

The algebraic expressions for these are easily given. For the propagator correction, we get

$$-i\Sigma(p^2) = -i\frac{\lambda_0}{2}\int\frac{d^4l}{(2\pi)^4}\frac{i}{l^2-\mu_0^2+i\epsilon} \tag{6}$$

Note that the integral is quadratically divergent.

The first of the three vertex diagrams gives

$$\Gamma_a = \Gamma(p) = \Gamma(s) = \frac{(-i\lambda_0)^2}{2}\int\frac{d^4l}{(2\pi)^4}\frac{i}{l^2-\mu_0^2+i\epsilon}\frac{i}{(l+p_1+p_2)^2-\mu_0^2+i\epsilon} \tag{7}$$

and it is obvious that

$$\Gamma_b = \Gamma(t)\,, \quad \Gamma_c = \Gamma(u)\,, \tag{8}$$

where we have defined

$$s=(p_1+p_2)^2\,,\quad t=(p_1-p_3)^2\,,\quad u=(p_1-p_4)^2\,. \tag{9}$$

The integral defining Γ is only logarithmically divergent.

There are also 1PI vertex corrections in which the bare vertex is corrected by inserting the 1PI correction into one of the external lines. These will be handled in a special manner later on.

We will typically try to separate "standard" divergent pieces from finite pieces according to some scheme.

For example, if we write

$$\Gamma(p^2)=a_0+a_1p^2+\ldots+\frac{1}{n!}a_n(p^2)+\ldots\,,\quad a_n=\frac{\partial^n}{\partial^n p^2}\Gamma(p^2)\bigg|_{p^2=0}\,, \tag{10}$$

then only a_0 is divergent. Thus, we can write

$$\Gamma(s)=\Gamma(0)+\widetilde{\Gamma}(s) \tag{11}$$

where $\widetilde{\Gamma}(s)$ is finite and $\widetilde{\Gamma}(0)=0$.

So, let us now turn to

Mass and wave-function renormalization

which involves the 1 particle propagator corrections.

In ϕ^4 theory at one-loop, there is a special feature that is not typical of other field theories or higher orders. This is that $\Sigma(p^2) = \Sigma(0)$ because of the fact that at one loop p^2 does not enter into the closed loop. (See earlier figure.)

More generally, we must write

$$\Sigma(p^2) = \Sigma(\mu^2) + (p^2 - \mu^2)\Sigma'(\mu^2) + \widetilde{\Sigma}(p^2) \tag{12}$$

where $\Sigma(\mu^2)$ is quadratically divergent and $\Sigma'(\mu^2)$ is logarithmically divergent, before regularization. Further, from the above definition, $\widetilde{\Sigma}(\mu^2) = 0$ and $\widetilde{\Sigma}'(\mu^2) = 0$ (i.e. $\widetilde{\Sigma} \propto (p^2 - \mu^2)^2 + \ldots$ — something that will be important later).

(At one-loop, $\Sigma'(p^2) = \widetilde{\Sigma}'(p^2) = 0$, but this is not the case once higher orders are considered.)

(Note that there is no term linear in p, which would be a linearly divergent term, since a term $\propto p^\mu$ would not be Lorentz invariant and since analyticity forbids Σ being a function of $\sqrt{p^2}$.)

Let us substitute the result Eq. (12) into our earlier expression for $\Delta(p^2)$, Eq. (5). We obtain

$$i\Delta(p^2) = \frac{i}{p^2 - \mu_0^2 - \Sigma(\mu^2) - (p^2 - \mu^2)\Sigma'(\mu^2) - \widetilde{\Sigma}(p^2) + i\epsilon} . \tag{13}$$

We next define μ^2, the physical pole mass squared by the consistency requirement

$$\mu^2 = \mu_0^2 + \Sigma(\mu^2) . \tag{14}$$

Then, $i\Delta(p^2)$ has a pole at $p^2 = \mu^2$ by construction, and it is the pole or singularity that is the physical definition of the physical mass of the particle.

We should notice that if $\Sigma(\mu^2)$ should one day be computable given a theory with full ultraviolet (large momentum) information, it would be possible to figure out what μ_0^2 corresponds to the observed physical pole mass-squared μ^2. But, even if this is not possible, we can still carry on in working out low-energy predictions of the theory so long as every such prediction can be rephrased in terms of the experimentally observed μ^2. Theories for which the latter is true define what we mean by a renormalizable theory or model.

Anyway, inserting the consistency requirement Eq. (14), we obtain

$$i\Delta(p^2) = \frac{i}{(p^2-\mu^2)[1-\Sigma'(\mu^2)] - \widetilde{\Sigma}(p^2) + i\epsilon}\,. \tag{15}$$

the infinity of $\Sigma(\mu^2)$ has been absorbed into the definition of the mass. Now we must turn to how to deal with $\Sigma'(\mu^2)$. We will discuss the procedure at order λ_0 (i.e. one-loop level). Both $\Sigma'(\mu^2)$ and $\widetilde{\Sigma}(p^2)$ are $\mathcal{O}(\lambda_0)$. Thus, perturbatively (we of course imagine that we have regulated any infinities, or, equivalently, that we know the full theory at high momentum and have simply input the true finite values) we can write

$$\widetilde{\Sigma}(p^2) \simeq [1-\Sigma'(\mu^2)]\widetilde{\Sigma}(p^2) + \mathcal{O}(\lambda_0^2)\,, \tag{16}$$

in which case we can (to order λ_0) write

$$i\Delta(p^2) = \frac{iZ_\phi}{p^2-\mu^2-\widetilde{\Sigma}(p^2)+i\epsilon}\,, \tag{17}$$

where

$$Z_\phi = \frac{1}{1-\Sigma'(\mu^2)} \simeq 1 + \Sigma'(\mu^2) + \mathcal{O}(\lambda_0^2)\,. \tag{18}$$

In this form, the remaining naively divergent component is a multiplicative factor and can be removed by redefining the field operator ϕ:

$$\phi = Z_\phi^{-1/2}\phi_0 \,. \tag{19}$$

We would then have the renormalized propagator

$$\begin{aligned} i\Delta_R(p^2) &\equiv \int d^4x e^{-ip\cdot x}\langle 0|T\{\phi(x)\phi(0)\}|0\rangle \\ &= Z_\phi^{-1}\int d^4x e^{-ip\cdot x}\langle T|\{\phi_0(x)\phi_0(0)\}|0\rangle \\ &= \frac{i}{p^2-\mu^2-\widetilde{\Sigma}(p^2)+i\epsilon} \\ &= iZ_\phi^{-1}\Delta(p^2)\,, \end{aligned} \tag{20}$$

which is now completely finite.

The quantity Z_ϕ is called the *wave function renormalization constant*. Renormalized Green's functions are more generally defined by

$$G_R^{(n)}(x_1,\ldots) = \langle 0|T\{\phi(x_1)\ldots\phi(x_n)\}|0\rangle$$

$$
\begin{aligned}
&= Z_\phi^{-n/2}\langle 0|T\{\phi_0(x_1)\ldots|0\rangle \\
&= Z_\phi^{-n/2} G_0^{(n)}(x_1,\ldots)\,. \qquad (21)
\end{aligned}
$$

One finds that the amplitudes for scattering of physical particles are given in terms of an appropriate projection operator operating on the appropriate G_R renormalized Green's function, and will thus be finite.

In momentum space, the same type of relationship between renormalized and bare Green's functions applies.

$$
G_R^{(n)}(p_1,\ldots,p_n) = Z_\phi^{-n/2} G_0^{(n)}(p_1,\ldots,p_n) \qquad (22)
$$

where

$$
(2\pi)^4\delta^4(p_1+\ldots+p_n)G_R^{(n)}(p_1,\ldots) = \int \prod_{i=1}^{n} d^4x_i e^{-ip_i\cdot x_i} G_R^{(n)}(x_1,\ldots)\,. \qquad (23)
$$

To go from the connected Green's function to the 1PI (amputated) Green's function, Γ, we remove the 1P reducible diagrams, and *also remove the propagators for the external lines*, **i.e. remove the $\Delta_R(p_i)$'s from $G_R^{(n)}(p_1\ldots p_n)$**

and $\Delta(p_i)$'s from $G_0^{(n)}(p_1 \ldots p_n)$. Algebraically, this means

$$G_0^{(n)}(p_1, \ldots) = \prod_{i=1}^{n} \Delta(p_i) \Gamma_0^{(n)}(p_1, \ldots),$$
$$G_R^{(n)}(p_1, \ldots) = \prod_{i=1}^{n} \Delta_R(p_i) \Gamma_R^{(n)}(p_1, \ldots). \quad (24)$$

Obviously, this gives us

$$\frac{\Gamma_R^{(n)}(p_1, \ldots)}{\Gamma_0^{(n)}(p_1, \ldots)} = \frac{G_R^{(n)}(p_1, \ldots)}{G_0^{(n)}(p_1, \ldots)} \frac{\prod_{i=1}^{n} \Delta(p_i)}{\prod_{i=1}^{n} \Delta_R(p_i)} = Z_\phi^{-n/2} Z_\phi^n = Z_\phi^{n/2}. \quad (25)$$

The quantity Z_ϕ is precisely the quantity Z, specialized to the ϕ^4 model, that we talked about earlier that relates (at asymptotic times) a field with full free-particle normalization to the fully interacting field that must also connect to multiple particle states and thus only has part of the full free-particle normalization at asymptotic times.

To see this, let us recall the way we introduced Z before when discussing the reduction formalism. In fact, let us first briefly review the reduction formalism for the scalar field.

The Reduction Formalism

Recall that we want to compute

$$ {}_{out}\langle p_1, p_2, \ldots | k_A, k_B \rangle_{in} \tag{26} $$

for a $2 \to n$ process, where the in and out states are eigenstates of the full H at times when we have highly separated wave packets. In such a distant past or future, the operator for creating a one particle wave packet should be free-particle like. Thus, we would, for instance, like to write

$$ |\vec{k}\rangle_{in} = a_{\vec{k}}^{in\,\dagger} |\Omega\rangle \tag{27} $$

where $|\Omega\rangle$ represents the fully interacting vacuum. Recall that the state created should be a free particle state but with the actually measured physical mass μ.

Next, we recall that for a normalization in which we write

$$ \phi_0 = \sum_{\vec{k}} \frac{1}{\sqrt{2VE_{\vec{k}}}} \left[a_{\vec{k}} e^{-ik\cdot x} + a_{\vec{k}}^{\dagger} e^{+ik\cdot x} \right] \tag{28} $$

we have

$$
\begin{aligned}
a^{in\,\dagger}_{\vec{k}} &= -i \int \frac{d^3\vec{x}}{\sqrt{2VE_{\vec{k}}}} \left[e^{-ik\cdot x} \overleftrightarrow{\partial}_{x_0} \phi_{in}(x^0, \vec{x}) \right] \\
a^{in}_{\vec{k}} &= +i \int \frac{d^3\vec{x}}{\sqrt{2VE_{\vec{k}}}} \left[e^{ik\cdot x} \overleftrightarrow{\partial}_{x_0} \phi_{in}(x^0, \vec{x}) \right] ,
\end{aligned}
\tag{29}
$$

in the case of a free field theory. Thus, it is not unreasonable to suppose that we should write something similar in the distant past or distant future even for the fully interacting ϕ_0. Thus, let us write

$$
|\vec{k}\rangle_{in} = -iZ^{-1/2} \lim_{x^0 \to -\infty} \int \frac{d^3\vec{x}}{\sqrt{2VE_{\vec{k}}}} \left[e^{-ik\cdot x} \overleftrightarrow{\partial}_{x_0} \phi_0(x) \right] |\Omega\rangle , \tag{30}
$$

where $Z^{-1/2}$ is the renormalization factor necessary to boost the interacting ϕ_0 to free-particle-like normalization. Another way of saying this is that

$$
\lim_{x^0 \to -\infty} \phi_0(x) = \sqrt{Z} \phi_{in}(x) . \tag{31}
$$

Once again, Z is required since $\phi_0(x)|\Omega\rangle$ in the interacting theory $\Rightarrow$ 1 particle, 2 particle, . . . states (i.e. it does much more than the ϕ_{in} operator which only makes 1 particle states). In fact though, Eq. (31) cannot really be true in a strict operator sense. The reason is that both ϕ_{in} and ϕ_0 are supposed to obey the same equal-time commutation relations

$$[\phi_{in}(x), \dot{\phi}_{in}(y)]_{[x^0=y^0]} = i\delta^3(\vec{x}-\vec{y})\,, \quad [\phi_0(x), \dot{\phi}_0(y)]_{[x^0=y^0]} = i\delta^3(\vec{x}-\vec{y})\,, \tag{32}$$

whereas the above relation would imply a factor of Z in front of the ϕ_0 commutation relation if the ϕ_{in} commutation relation holds. Fortunately, the real requirement is not so strict. What we need is only that

$$\begin{aligned} \lim_{x^0\to-\infty} \phi_0^-(x)|\ldots\rangle_{in} &= \sqrt{Z}\phi_{in}^-(x)|\ldots\rangle_{in}\,, \\ \lim_{x^0\to+\infty} {}_{out}\langle\ldots|\phi_0^+(x) &= \sqrt{Z}\;{}_{out}\langle\ldots|\phi_{out}^+(x)\,, \end{aligned} \tag{33}$$

where the $+,-$ superscripts are the positive and negative frequency components of the fields containing the $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ operators respectively. The above is all that is required since all we really need to do at very early or very late times is to be able create appropriate stuff in the incoming state or outgoing state (the latter having been written in the conjugate form appropriate to an S matrix calculation). Since no statement is made above about the positive frequency components, there is no contradiction with relations between commutators which require knowing how $+$ and $-$ frequency components commute. We will return later to see that this $\sqrt{Z}$ factor must be the same as that which appeared in our 1PI etc. sum to get the full propagator.

Now let us look at the reduction process. We have

$$
\begin{aligned}
{}_{out}\langle p_1, p_2, \ldots | k_A, k_B\rangle_{in} &= {}_{out}\langle p_1, p_2, \ldots | a^{in\,\dagger}_{\vec{k}_A} | k_B\rangle_{in} \\
&= -iZ^{-1/2} \lim_{x^0_A \to -\infty} \int_{x^0_A} \frac{d^3\vec{x}_A}{\sqrt{2VE_{\vec{k}_A}}} \left[e^{-ik_A\cdot x_A} \overleftrightarrow{\partial}_{x^0_A} {}_{out}\langle p_1 \ldots | \phi_0(x_A) | k_B\rangle_{in} \right]
\end{aligned} \tag{34}
$$

At this point, we bring in the trivial identity

$$
\left[\lim_{t\to+\infty} - \lim_{t\to-\infty}\right] \int d^3\vec{x}\,\psi(\vec{x},t) = \lim_{t_f\to\infty, t_i\to-\infty} \int_{t_i}^{t_f} dt \frac{\partial}{\partial t} \int d^3\vec{x}\,\psi(\vec{x},t)\,. \tag{35}
$$

We use this to substitute in the above equation, obtaining

$$
\begin{aligned}
{}_{out}\langle p_1, p_2, \ldots | k_A, k_B\rangle_{in} &= {}_{out}\langle p_1, p_2, \ldots | a^{out\,\dagger}_{\vec{k}_A} | k_B\rangle_{in} \\
&\quad + iZ^{-1/2} \int \frac{d^4x_A}{\sqrt{2VE_{\vec{k}_A}}} \partial_{x^0_A} \left[e^{-ik_A\cdot x_A} \overleftrightarrow{\partial}_{x^0_A} {}_{out}\langle p_1 \ldots | \phi_0(x_A) | k_B\rangle_{in} \right]
\end{aligned}
$$

$$= 0 + iZ^{-1/2} \int \frac{d^4x_A}{\sqrt{2VE_{\vec{k}_A}}} \left[e^{-ik_A \cdot x_A} \partial^2_{x^0_A} \, {}_{out}\langle p_1 \ldots | \phi_0(x_A) | k_B \rangle_{in} \right.$$

$$\left. - \partial^2_{x^0_A} e^{-ik_A \cdot x_A} \, {}_{out}\langle p_1 \ldots | \phi_0(x_A) | k_B \rangle_{in} \right]$$

$$= iZ^{-1/2} \int \frac{d^4x_A}{\sqrt{2VE_{\vec{k}_A}}} e^{-ik_A \cdot x_A} (\Box_{x_A} + \mu^2) \, {}_{out}\langle p_1 \ldots | \phi_0(x_A) | k_B \rangle_{in} \quad (36)$$

where to obtain the last line we noted that

$$\partial^2_{x^0_A} e^{-ik_A \cdot x_A} = \left(\vec{\nabla}^2_{\vec{x}_A} - \mu^2 \right) e^{-ik_A \cdot x_A} \, , \quad (37)$$

substituted it in, and partial integrated twice.

Now, we reduce in a 2nd particle, let us say the p_1 outgoing particle. We have

$$\begin{aligned} {}_{out}\langle p_1 \ldots | \phi_0(x_A) | k_B \rangle_{in} &= {}_{out}\langle p_2 \ldots | a_{out}(\vec{p}_1) \phi_0(x_A) | k_B \rangle_{in} \\ &= \lim_{y^0_1 \to \infty} iZ^{-1/2} \int \frac{d^3y_1}{\sqrt{2VE_{\vec{p}_1}}} \left[e^{ip_1 \cdot y_1} \overleftrightarrow{\partial}_{y^0_1} \, {}_{out}\langle p_2 \ldots | \phi_0(y_1) \phi_0(x_A) | k_B \rangle_{in} \right] \\ &= \lim_{y^0_1 \to \infty} iZ^{-1/2} \int \frac{d^3y_1}{\sqrt{2VE_{\vec{p}_1}}} \left[e^{ip_1 \cdot y_1} \overleftrightarrow{\partial}_{y^0_1} \, {}_{out}\langle p_2 \ldots | T\{\phi_0(y_1) \phi_0(x_A)\} | k_B \rangle_{in} \right] \end{aligned}$$

$$= \lim_{y_1^0 \to -\infty} iZ^{-1/2} \int \frac{d^3 y_1}{\sqrt{2VE_{\vec{p}_1}}} \left[e^{ip_1 \cdot y_1} \overleftrightarrow{\partial}_{y_1^0} \, {}_{out}\langle p_2 \ldots | T\{\phi_0(y_1)\phi_0(x_A)\} | k_B \rangle_{in} \right] + correction$$

$$= {}_{out}\langle p_2 \ldots | \phi_0(x_A) a_{in}(\vec{p}_1) | k_B \rangle_{in} + correction$$

$$= 0 + correction \,, \qquad (38)$$

where we assumed that $p_1 \neq k_B$. Writing out the correction piece, we thus have

$$\begin{aligned} {}_{out}\langle p_1 \ldots | \phi_0(x_A) | k_B \rangle_{in} &= iZ^{-1/2} \int \frac{d^4 y_1}{\sqrt{2VE_{\vec{p}_1}}} \partial_{y_1^0} \left[e^{ip_1 \cdot y_1} \overleftrightarrow{\partial}_{y_1^0} \, {}_{out}\langle p_2 \ldots | T\{\phi_0(y_1)\phi_0(x_A)\} | k_B \rangle_{in} \right] \\ &= iZ^{-1/2} \int \frac{d^4 y_1}{\sqrt{2VE_{\vec{p}_1}}} e^{ip_1 \cdot y_1} (\Box_{y_1} + \mu^2) \, {}_{out}\langle p_2 \ldots | T\{\phi_0(y_1)\phi_0(x_A)\} | k_B \rangle_{in} \end{aligned} \qquad (39)$$

where we did the same manipulations as in the previous case, i.e. employed $(\Box_{y_1} + \mu^2) e^{-ip_1 \cdot y_1} = 0$ to replace

$$\partial^2_{y_1^0} e^{-ip_1 \cdot y_1} = (\vec{\nabla}^2_{\vec{y}_1} - \mu^2) e^{-ip_1 \cdot y_1} \qquad (40)$$

and then partial integrated twice in $\vec{y}_1$.

So, we would now insert this expression into Eq. (36). And, we would continue on to reduce in all the initial and final particles. I hope you can see that the result will be (for n final particles)

$$
\begin{aligned}
{}_{out}\langle p_1, p_2, \ldots | k_A, k_B \rangle_{in} &= [iZ^{-1/2}]^{n+2} \prod_{i=1}^{n+2} \frac{1}{\sqrt{2VE_i}} \int d^4y_1 \ldots d^4y_n d^4x_A d^4x_B \\
&\times \exp\left[i \sum_{k=1}^{n} p_k \cdot y_k - ik_A \cdot x_A - ik_B \cdot x_B \right] \\
&\times (\Box_{y_1} + \mu^2) \ldots (\Box_{x_B} + \mu^2) \langle \Omega | T\{\phi_0(y_1) \ldots \phi_0(x_B)\} | \Omega \rangle \,. \quad (41)
\end{aligned}
$$

Before going back to renormalization, let's just check that we get the correct Feynman rule. For the basic connected vertex graph, our Feynman rules give (we set $Z = 1$ for this tree level case)

$$
\begin{aligned}
\langle \Omega | T\{\phi_0(x_1) \ldots \phi_0(x_4)\} | \Omega \rangle &\equiv G_0^{(4)}(x_1, x_2, x_3, x_4) \\
&= \left[\prod_i \int \frac{d^4 l_i}{(2\pi)^4} \frac{i}{l_i^2 - \mu^2 + i\epsilon} \right] (-i\lambda)(2\pi)^4 \delta^4(\sum_j l_j) e^{-i \sum_k l_k \cdot x_k} \,. \quad (42)
\end{aligned}
$$

We now insert this into the above expression for the S matrix, but using the convention where all particles are outgoing. We find (all indexed sums and products go from 1 to 4):

$$
{}_{out}\langle p_1, p_2, p_3, p_4 | 0 \rangle_{in} = [i]^4 \left[\prod_m \frac{1}{\sqrt{2VE_m}} \right] \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{i \sum p_k \cdot x_k}
$$

$$
\begin{aligned}
& \times(\Box_{x_1}+\mu^2)\ldots(\Box_{x_4}+\mu^2)G_0^{(4)}(x_1,x_2,x_3,x_4) \\
= \quad & [i]^4\left[\prod_m \frac{1}{\sqrt{2VE_m}}\right]\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{i\sum p_k\cdot x_k} \\
& \times\left[\prod_i \int \frac{d^4l_i}{(2\pi)^4}(-l_i^2+\mu^2)\frac{i}{l_i^2-\mu^2+i\epsilon}\right](-i\lambda)(2\pi)^4\delta^4(\sum_j l_j)e^{-i\sum_k l_k\cdot x_k} \\
= \quad & \left[\prod_m \frac{1}{\sqrt{2VE_m}}\right]\left[\prod_i \int \frac{d^4l_i}{(2\pi)^4}(-i)(2\pi)^4\delta^4(p_i-l_i)\right](-i\lambda)(2\pi)^4\delta^4(\sum_j l_j) \\
= \quad & \left[\prod_m \frac{1}{\sqrt{2VE_m}}\right](-i\lambda)(2\pi)^4\delta^4(\sum_j p_j)\,. \qquad (43)
\end{aligned}
$$

As always, we remove the $\frac{1}{\sqrt{2VE}}$ factors and the $(2\pi)^4\delta^4(\ldots)$ to define $i\mathcal{M}$ and obtain at tree-level

$$\mathcal{M} = -\lambda\,. \qquad (44)$$

Let us generalize this a bit. We will write

$$G_0(x_1,\ldots,x_n) = \int \frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}\exp[i\sum p_j\cdot x_j]\widetilde{G}_0(p_1,\ldots)\,. \qquad (45)$$

Translational invariance implies that

$$\widetilde{G}_0(p_1,\ldots) \quad = \quad (2\pi)^4\delta^4(p_1+\ldots+p_n)G_0(p_1,\ldots)$$

$$= \int d^4x_1 \ldots d^4x_n \exp[-i \sum p_k \cdot x_k] G_0(x_1, \ldots) \quad (46)$$

where the 2nd line gives the inverse Fourier transform.

Thus, the S matrix structure in the general case looks like

$$\begin{aligned}
{}_{out}\langle p_1, \ldots p_n | \Omega \rangle_{in} &= (iZ^{-1/2})^n \int d^4x_1 \ldots d^4x_n \exp[-i \sum_k p_k \cdot x_k](\Box_{x_1} + \mu^2) \ldots (\Box_{x_n} + \mu^2) G_0(x_1 \ldots x_n) \\
&= (iZ^{-1/2})^n \int d^4x_1 \ldots d^4x_n \exp[-i \sum_k p_k \cdot x_k](\Box_{x_1} + \mu^2) \ldots (\Box_{x_n} + \mu^2) \\
&\quad \times \int \frac{d^4\bar{p}_1}{(2\pi)^4} \ldots \exp[i \sum \bar{p}_i \cdot x_i](2\pi)^4 \delta^4(\bar{p}_1 + \ldots + \bar{p}_n) G_0(\bar{p}_1 \ldots \bar{p}_n) \\
&= (iZ^{-1/2})^n \int d^4\bar{p}_1 \ldots d^4\bar{p}_n \delta^4(p_1 - \bar{p}_1) \ldots \delta^4(p_n - \bar{p}_n)(\mu^2 - \bar{p}_1^2) \ldots (\mu^2 - \bar{p}_n^2) \\
&\quad \times (2\pi)^4 \delta^4(\bar{p}_1 + \ldots + \bar{p}_n) G_0(\bar{p}_1, \ldots \bar{p}_n) \\
&= (iZ^{-1/2})^n \int d^4\bar{p}_1 \ldots d^4\bar{p}_n \delta^4(p_1 - \bar{p}_1) \ldots \delta^4(p_n - \bar{p}_n)(\mu^2 - \bar{p}_1^2) \ldots (\mu^2 - \bar{p}_n^2) \\
&\quad \times (2\pi)^4 \delta^4(\bar{p}_1 + \ldots + \bar{p}_n) \frac{iZ}{\bar{p}_1^2 - \mu^2 - \widetilde{\Sigma}(\bar{p}_1^2)} \ldots \frac{iZ}{\bar{p}_n^2 - \mu^2 - \widetilde{\Sigma}(\bar{p}_n^2)} \Gamma_0(\bar{p}_1, \ldots \bar{p}_n)
\end{aligned} \quad (47)$$

where we wrote

$$G_0(\bar{p}_1\ldots) = \left[\prod_i i\Delta(\bar{p}_i^2)\right]\Gamma_0 \qquad (48)$$

with the $i\Delta(\bar{p}_i^2)$ factors coming from the full insertion set on each of the external lines (see figure).

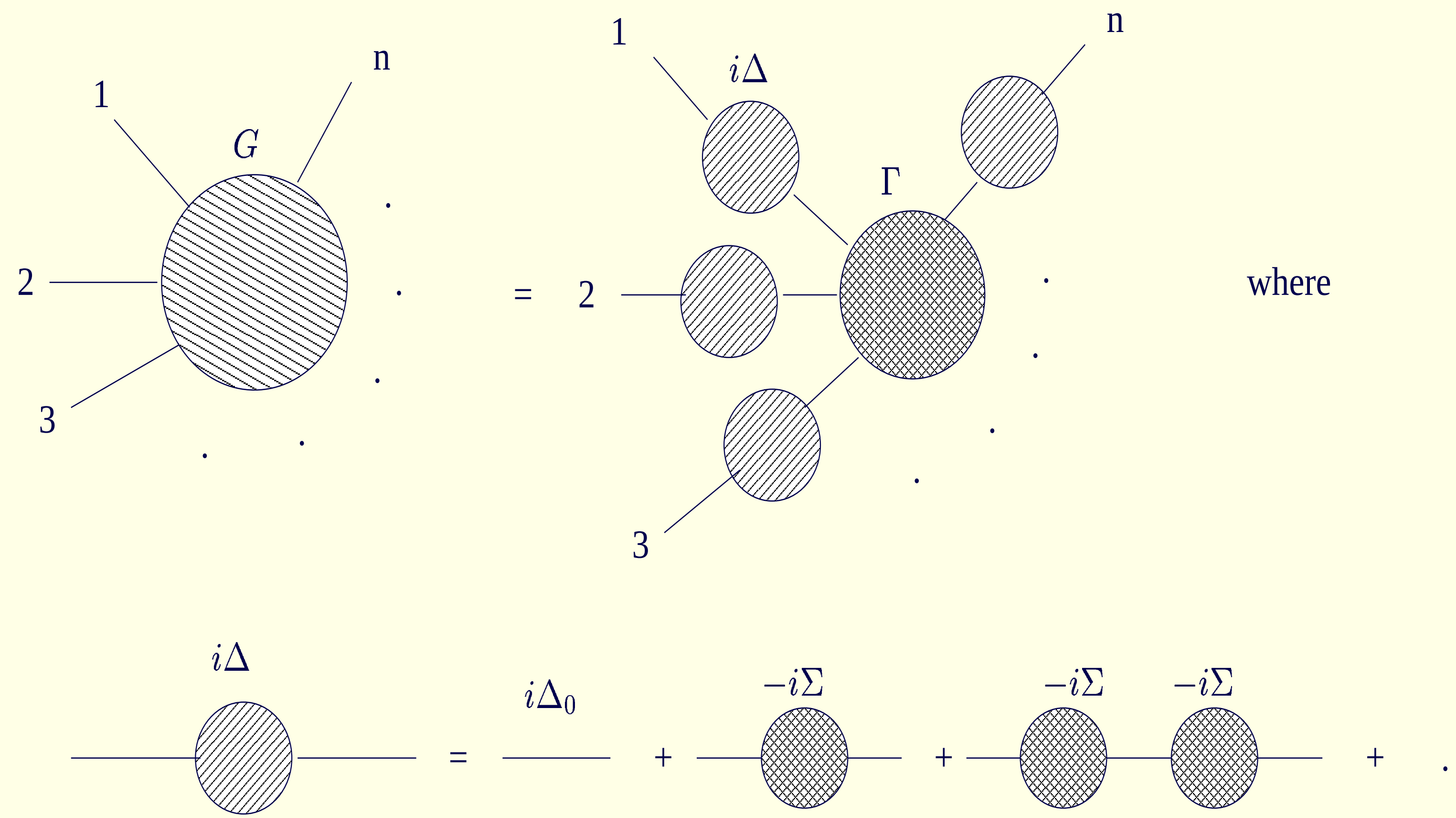


Figure 2: Graphical representation of Eq. (48).

Now, as we go on shell (which is what happens since $\bar{p}_i = p_i$ via the δ^4 functions and since $p_i^2 = \mu^2$) we have

$$(\mu^2 - \bar{p}_i^2)\frac{iZ}{\bar{p}_i^2 - \mu^2 - \widetilde{\Sigma}(\bar{p}_i^2)} \to -iZ \tag{49}$$

for each i, leaving us with

$$\begin{aligned} {}_{out}\langle p_1, \ldots p_n|\Omega\rangle_{in} &= (iZ^{-1/2})^n(-iZ)^n(2\pi)^4\delta^4(p_1 + \ldots + p_n)\Gamma_0(p_1 \ldots) \\ &= (Z^{1/2})^n(2\pi)^4\delta^4(p_1 + \ldots + p_n)\Gamma_0(p_1 \ldots) \\ &= (2\pi)^4\delta^4(p_1 + \ldots + p_n)\Gamma_R(p_1 \ldots)\,, \end{aligned} \tag{50}$$

where we used Eq. (25). Thus, you see that the S matrix elements are given in terms of the renormalized "amputated" momentum space n-particle interaction. This is the key result we need. We need only show now that Γ_R can be written as a finite expression in terms of a single (finite) measurement of the 4-point interaction at some conveniently chosen momentum setup.

But first, we want to return to the claim that the Z_ϕ that appears in

$$i\Delta(p^2) = \frac{iZ_\phi}{p^2 - \mu^2 - \widetilde{\Sigma}(p^2) + i\epsilon} \tag{51}$$

is the same Z as that which appeared in our early and late time limits of the ϕ_0 field versus the $\phi_{in,out}$ fields. Obviously, the cancellation of all the Z's in the end will only occur if this is the case.

Without getting too formal about it (for more rigor, see Peskin), that these two Z's are the same is certainly intuitively obvious. We wrote $Z^{-1/2}\phi_0^-|in\rangle \rightarrow \phi_{in}^-|in\rangle$ at very early times and $Z^{-1/2}\langle out|\phi_0^+ \rightarrow \langle out|\phi_{out}^+$ at very late times. Here, ϕ_0 is the fully interacting field while the out and in fields are those that are free particle like with full one-particle normalization at very early or very late times. It is the latter that are related to the a^{out} and a^{in} operators that define the S-matrix.

Consider the propagator for $p^2 \rightarrow \mu^2$. For the single particle propagator, the distinction between in and out states is not relevant since interactions that change the state will not be included. We have

$$\int d^4x e^{-ip\cdot(x-y)}\langle 0|T\{\phi_0(x)\phi_0(y)\}|0\rangle = \frac{iZ_\phi}{p^2-\mu^2}. \quad (52)$$

Focusing on the $\langle 0|T\{\phi_0(x)\phi_0(y)\}|0\rangle$ part above, let us consider $x^0 \rightarrow +\infty$ and $y^0 \rightarrow -\infty$. Then,

$$\lim_{x^0\rightarrow\infty, y^0\rightarrow-\infty} \langle\Omega|T\{\phi_0(x)\phi_0(y)\}|\Omega\rangle$$

$$
\begin{aligned}
&= \langle\Omega|\phi_0(x)\phi_0(y)|\Omega\rangle \\
&= \langle\Omega|\phi_0^+(x)\phi_0^-(y)|\Omega\rangle \\
&= \langle\Omega|Z^{1/2}\phi_{out}^+ Z^{1/2}\phi_{in}^-|\Omega\rangle \\
&= Z\langle\Omega|\phi_{out}^+\phi_{in}^-|\Omega\rangle\,. \qquad (53)
\end{aligned}
$$

Remembering that I insisted at the time that the in and out fields be free-particle like, but with physical mass μ, the resulting contribution to the propagator will be with unit normalization, yielding a propagator after Fourier transforming that is $\frac{i}{p^2-\mu^2}$ as $p^2 \to \mu^2$. This makes it clear that the Z factor that appears in the asymptotic time relations must be the same as the Z_ϕ which appears in the propagator. Note that it is important in making the above switch to the Fourier transform to emphasize that the limit of $p^2 \to \mu^2$ means that we are really isolating just the free-particle-like component of the interacting field which connects a state very early in time to one very late in time. This means that even though I wrote things in the $\lim_{x^0\to\infty, y^0\to-\infty}$, this was not actually necessary in the $p^2 \to \mu^2$ limit. Further, there is no difference between ϕ_{in} and ϕ_{out} in the $p^2 \to \mu^2$ limit. Physically, all the above statements are equivalent to the statements that:

- one-particle states are stable, i.e. $|1\rangle_{in} = |1\rangle_{out}$;

- the vacuum is unique, $|\Omega\rangle_{in} = |\Omega\rangle_{out}$ (up the a trivial phase that is taken to be 1 by convention).

Thus, the full Fourier transform in this limit will have the Z factor relative to the canonical free-particle $i/(p^2 - \mu^2)$ form.

Still more on Z: the spectral representation

From the 2nd quantization expansion of ϕ_{in} and ϕ_0, it should be clear that $\langle 1|\phi_{in}|\Omega\rangle$ and $\langle 1|\phi_0|\Omega\rangle$ have the same functional dependence on x. The normalization factor $Z^{1/2}$ takes into account the fact that the content of the state $\phi_0(x)|\Omega\rangle$ is not exhausted by the matrix elements $\langle 1|\phi_0|\Omega\rangle$ whereas the state $\phi_{in}(x)|\Omega\rangle$ *is* exhausted by $\langle 1|\phi_{in}|\Omega\rangle$. Once again, this intuitively means that $Z^{1/2}$ should obey $0 \leq Z^{1/2} < 1$.

For a real field, using translation invariance (i.e. $\phi(x) = e^{iP\cdot x}\phi(0)e^{-iP\cdot x}$), $P|\Omega\rangle = \langle\Omega|P = 0$, $P|\alpha\rangle = p_\alpha|\alpha\rangle$, etc., we can write

$$\begin{aligned}\langle\Omega|[\phi_0(x), \phi_0(y)]|\Omega\rangle &= \sum_\alpha [\langle\Omega|\phi_0(0)|\alpha\rangle e^{-ip_\alpha\cdot(x-y)}\langle\alpha|\phi_0(0)|\Omega\rangle - (x \leftrightarrow y)] \\ &= \sum_\alpha |\langle\Omega|\phi_0(0)|\alpha\rangle|^2 \left[e^{-ip_\alpha\cdot(x-y)} - e^{+ip_\alpha\cdot(x-y)}\right], \quad (54)\end{aligned}$$

where the sum runs over a complete set of positive energy states α. To compare this with the commutator of two free fields of mass μ,

$$\Delta_0(x-y,\mu) \equiv [\phi_0^{free}(x),\phi_0^{free}(y)] = \int \frac{d^3l}{(2\pi)^3}\frac{1}{2E_{\vec{l}}}\left[e^{-il\cdot(x-y)} - e^{il\cdot(x-y)}\right]$$
$$= \int \frac{d^4l}{(2\pi)^3}\delta(l^2-\mu^2)\theta(l_0)\left[e^{-il\cdot(x-y)} - e^{+il\cdot(x-y)}\right] , \quad (55)$$

we insert into Eq. (54) the identity

$$1 = \int d^4l\delta^4(l-p_\alpha) \quad (56)$$

leading to

$$\langle\Omega|[\phi_0(x),\phi_0(y)]|\Omega\rangle = \int \frac{d^4l}{(2\pi)^3}\rho(l)\left(e^{-il\cdot(x-y)} - e^{+il\cdot(x-y)}\right) \quad (57)$$

with

$$\rho(l) = (2\pi)^3\sum_\alpha \delta^4(l-p_\alpha)\left|\langle\Omega|\phi_0(0)|\alpha\rangle\right|^2 . \quad (58)$$

Clearly, $\rho(l) > 0$ and vanishes when l is not in the forward light cone. Further, it is invariant under a Lorentz transformation, as required by the corresponding property of the field ϕ_0. Thus, we write

$$\rho(l) = \sigma(l^2)\theta(l^0)\,, \quad \text{with } \sigma(l^2) = 0 \text{ if } l^2 < 0\,. \tag{59}$$

In general, this is a positive measure where δ-function singularities might possibly occur. We shall also use the identity

$$\sigma(l^2) = \int d\mu'^2 \sigma(\mu'^2)\delta(l^2 - \mu'^2)\,. \tag{60}$$

Using the above, we have

$$\begin{aligned}
&\langle\Omega|[\phi_0(x), \phi_0(y)]|\Omega\rangle \\
&\quad = \int \frac{d^4 l}{(2\pi)^3}\rho(l)\left(e^{-il\cdot(x-y)} - e^{+il\cdot(x-y)}\right) \\
&\quad = \int \frac{d^4 l}{(2\pi)^3}\sigma(l^2)\theta(l^0)\left(e^{-il\cdot(x-y)} - e^{+il\cdot(x-y)}\right) \\
&\quad = \int d\mu'^2\sigma(\mu'^2)\int \frac{d^4 l}{(2\pi)^3}\delta(l^2 - \mu'^2)\theta(l^0)\left(e^{-il\cdot(x-y)} - e^{+il\cdot(x-y)}\right) \\
&\quad = \int d\mu'^2\sigma(\mu'^2)\Delta_0(x - y; \mu')
\end{aligned} \tag{61}$$

where we used the analogue of Eq. (55):

$$\Delta_0(x-y;\mu') \;=\; \int \frac{d^4l}{(2\pi)^3}\delta(l^2-\mu'^2)\theta(l^0)\left(e^{-il\cdot(x-y)}-e^{+il\cdot(x-y)}\right) \tag{62}$$

where, to repeat, $\Delta_0(x-y;\mu')$ is our old way of writing the free-field commutator. We now wish to separate out the 1-particle state component using

$$\langle 1|\phi_0|\Omega\rangle = Z^{1/2}\langle 1|\phi^{free}|\Omega\rangle = Z^{1/2}\langle 1|\phi^{in}|\Omega\rangle = Z^{1/2}\langle 1|\phi^{out}|\Omega\rangle\,. \tag{63}$$

For this, it is best to convert from finite volume normalization, for which $\sum_\alpha^{1\ particle}=\sum_{\vec k}$, to continuum $V\to\infty$ normalization using our old friend

$$\sum_{\vec k}\to\frac{d^3\vec k V}{(2\pi)^3} \tag{64}$$

and recall that for

$$\phi_0^{free}=\sum_{\vec q}\frac{1}{\sqrt{2VE_{\vec q}}}(a_{\vec q}^{free}e^{-iq\cdot x}+a_{\vec q}^{\dagger\,free}e^{iq\cdot x}) \tag{65}$$

and one particle states (with full normalization) defined by $|\vec{k}\rangle = a_{\vec{k}}^{\dagger\, free}|\Omega\rangle$, we have

$$\langle\Omega|\phi_0^{free}|\vec{k}\rangle = \frac{1}{\sqrt{2VE_{\vec{k}}}}e^{ik\cdot x}\,, \tag{66}$$

and, hence,

$$\langle\Omega|\phi_0^{interacting}|\vec{k}\rangle = Z^{1/2}\frac{1}{\sqrt{2VE_{\vec{k}}}}e^{ik\cdot x}\,, \tag{67}$$

where $E_{\vec{k}}$ is computed using μ^2. As a result, the one-particle content of ρ is computed as

$$\begin{aligned}
\rho(l) &= (2\pi)^3\sum_{\vec{k}}\delta^4(l-k)Z\frac{1}{2VE_{\vec{k}}} \\
&= (2\pi)^3\int\frac{d^3\vec{k}V}{(2\pi)^3}\delta^4(l-k)Z\frac{1}{2VE_{\vec{k}}} \\
&= Z\int\frac{d^3\vec{k}}{2E_{\vec{k}}}\delta^4(l-k) \\
&= Z\int d^4k\delta(k^2-\mu^2)\theta(k^0)\delta^4(l-k)
\end{aligned}$$

$$= \quad Z\delta(l^2 - \mu^2)\theta(l^0)\,, \tag{68}$$

from which we read off that

$$\sigma^{1\ particle}(l^2) = Z\delta(l^2 - \mu^2)\,, \tag{69}$$

given that $\rho(l) \equiv \sigma(l^2)\theta(l^0)$.

We now use Eq. (61) to conclude that

$$\langle\Omega|[\phi_0(x), \phi_0(y)]|\Omega\rangle = Z\Delta_0(x-y;\mu) + \int_{\mu_1^2} d\mu'^2 \sigma(\mu'^2)\Delta_0(x-y;\mu')\,, \tag{70}$$

where $\mu_1 > \mu$ is the threshold for multiparticle states ($= 3\mu$ in the ϕ^4 theory context — can't go from 1 to 2 particle state in ϕ^4). Now, assuming that the interaction Lagrangian does not involve field derivatives, $\dot{\phi}_0$ will be conjugate to ϕ_0. By taking the time derivative in x^0 of both sides of the above equation, and identifying the coefficients of $i\delta^3(x-y)$, we find

$$1 = Z + \int_{\mu_1^2} d\mu'^2 \sigma(\mu'^2)\,, \tag{71}$$

so that $\sigma > 0$ implies $0 \leq Z \leq 1$.

For the time-ordered product, we follow a very similar procedure. We start with

$$\langle\Omega|\phi_0(x)\phi_0(y)|\Omega\rangle = \sum_\alpha \langle\Omega|\phi_0(0)|\alpha\rangle e^{-ip_\alpha\cdot(x-y)}\langle\alpha|\phi_0(0)|\Omega\rangle\,, \quad (72)$$

and introduce

$$1 = \int d^4l\delta^4(l - p_\alpha) \quad (73)$$

leading to

$$\langle\Omega|\phi_0(x)\phi_0(y)|\Omega\rangle = \int \frac{d^4l}{(2\pi)^3}\rho(l)e^{-il\cdot(x-y)} \quad (74)$$

with $\rho(l)$ defined as before. Using this, we can write

$$\begin{aligned}
\theta(x^0 - y^0)\langle\Omega|\phi_0(x)\phi_0(y)|\Omega\rangle &= \theta(x^0 - y^0)\int \frac{d^4l}{(2\pi)^3}\sigma(l^2)\theta(l^0)e^{-il\cdot(x-y)} \\
&= \theta(x^0 - y^0)\int d\mu'^2 \int \frac{d^4l}{(2\pi)^3}\delta(l^2 - \mu'^2)\theta(l^0)e^{-il\cdot(x-y)}\sigma(\mu'^2) \\
&= \theta(x^0 - y^0)\int d\mu'^2 \int \frac{d^3l}{(2\pi)^3}\frac{1}{2E_{\vec{l}}}e^{-il\cdot(x-y)}\sigma(\mu'^2)\,.
\end{aligned}$$

$$= \quad \theta(x^0 - y^0) \int d\mu'^2 \int \frac{d^4 l}{(2\pi)^4} \frac{i}{l^2 - \mu'^2 + i\epsilon} e^{-il\cdot(x-y)} \sigma(\mu'^2) \,. \tag{75}$$

(For $x^0 > y^0$, the dl^0 contour is closed down and is clockwise, picking up the $l^0 = +E_{\vec{l}}$ pole in the lower 1/2 plane.) Similarly, if $y^0 > x^0$ we have

$$\theta(y^0 - x^0)\langle\Omega|\phi_0(y)\phi_0(x)|\Omega\rangle \quad = \quad \theta(y^0 - x^0) \int d\mu'^2 \int \frac{d^4 l}{(2\pi)^4} \frac{i}{l^2 - \mu'^2 + i\epsilon} e^{-il\cdot(x-y)} \sigma(\mu'^2) \,. \tag{76}$$

Combining, we obviously obtain (using $\theta(x^0 - y^0) + \theta(y^0 - x^0) = 1$)

$$\langle\Omega|T\{\phi_0(x)\phi_0(y)\}|\Omega\rangle = \int d\mu'^2 \Delta_F(x - y, \mu')\sigma(\mu'^2) \,, \tag{77}$$

where Δ_F is the usual Feynman propagator. We now take the Fourier transform of this result

$$\begin{aligned}
\int d^4 x e^{-ip\cdot x}\langle\Omega|T\{\phi_0(x)\phi_0(0)\}|\Omega\rangle &= \int d\mu'^2 \frac{i}{p^2 - \mu'^2 + i\epsilon}\sigma(\mu'^2) \\
&= \frac{iZ}{p^2 - \mu^2 + i\epsilon} + \int_{9\mu^2} d\mu'^2 \frac{i}{p^2 - \mu'^2 + i\epsilon}\sigma(\mu'^2) \\
&= \frac{iZ_\phi}{p^2 - \mu^2 - \widetilde{\Sigma}(p^2) + i\epsilon}
\end{aligned} \tag{78}$$

where we isolated the 1-particle portion of σ, which explicitly contains the Z factor defined by asymptotic state limits. Now, as $p^2 \to \mu^2$, the $\int_{9\mu^2}$ term above has no singularity and $\widetilde{\Sigma}(p^2) \propto (p^2-\mu^2)^2$ (by its definition) and so we conclude that $Z_\phi = Z$.

Coupling Constant Renormalization

We consider the 1PI 4-point function at one-loop. The expression for it is

$$\Gamma_0^{(4)}(s,t,u) = -i\lambda_0 + \Gamma(s) + \Gamma(t) + \Gamma(u)\,. \tag{79}$$

Noting that $s+t+u = 4\mu^2$ we choose our experimental input observation as that made at the (unphysical, but symmetric) point

$$s_0 = t_0 = u_0 = \frac{4\mu^2}{3}\,. \tag{80}$$

We define the renormalized coupling constant (the thing we measure) by

$$\Gamma_R^{(4)}(s_0,t_0,u_0) \equiv -i\lambda\,. \tag{81}$$

We next write

$$\Gamma_0^{(4)}(s,t,u) = -i\lambda_0 + 3\Gamma(s_0) + \widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)\,, \tag{82}$$

where $\widetilde{\Gamma}(s) \equiv \Gamma(s) - \Gamma(s_0) \to 0$ when $s = s_0$. Now define

$$-\,iZ_\lambda^{-1}\lambda_0 = -i\lambda_0 + 3\Gamma(s_0) \tag{83}$$

so that

$$\Gamma_0^{(4)}(s,t,u) = -iZ_\lambda^{-1}\lambda_0 + \widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u) \tag{84}$$

and

$$\Gamma_0^{(4)}(s_0,t_0,u_0) = -iZ_\lambda^{-1}\lambda_0\,. \tag{85}$$

Z_λ carries the infinities.

Now, we use the relation

$$\Gamma_R^{(4)}(s,t,u) = Z_\phi^2\Gamma_0^{(4)}(s,t,u) \tag{86}$$

discussed earlier, $\Gamma_R^{(4)}(s_0,t_0,u_0) \equiv -i\lambda$, and the above equations to obtain

$$\lambda = Z_\phi^2 Z_\lambda^{-1}\lambda_0\,. \tag{87}$$

We can now show that the renormalized 1PI four-point function will be finite up to order λ^3 when written in terms of λ. This follows from

$$\begin{aligned}
\Gamma_R^{(4)}(p_1,\ldots p_4) &= Z_\phi^2 \Gamma_0^{(4)}(p_1\ldots) \\
&= -iZ_\lambda^{-1} Z_\phi^2 \lambda_0 + Z_\phi^2 \left[\widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)\right] \\
&= -i\lambda + Z_\phi^2 \left[\widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)\right] \\
&\simeq -i\lambda + \left[\widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)\right] + \mathcal{O}\left(\lambda^3\right)
\end{aligned} \tag{88}$$

given that $Z_\phi = 1 + \mathcal{O}(\lambda_0)$ and $\widetilde{\Gamma} = \mathcal{O}(\lambda_0^2)$. Clearly, to the order considered, the final expression for $\Gamma_R^{(4)}$ above is completely finite when expressed as a function of the renormalized (i.e. measured) coupling λ.

Let us look at $G^{(4)}$ in very explicit fashion using this same approach. To order λ^2, we need to consider the bubble insertions on external propagators as well as the 1PI fish diagrams. There are 4 such bubble insertions. We can write

$$G_0^{(4)}(p_1,\ldots p_4) \quad = \quad \prod_j \left(\frac{i}{p_j^2 - \mu_0^2 + i\epsilon}\right) \Bigg[-i\lambda_0 + 3\Gamma(s_0) + \widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)$$

$$+(-i\lambda_0)\sum_k [-i\Sigma(p_k^2)]\frac{i}{p_k^2-\mu_0^2+i\epsilon}\Bigg] \,. \tag{89}$$

The first and last terms can be combined *to order* λ_0^2 **in the form:**

$$(-i\lambda_0)\left[\prod_j\left(\frac{i}{p_j^2-\mu_0^2+i\epsilon}\right)\right]\left[1+\sum_k \Sigma(p_k^2)\frac{1}{p_k^2-\mu_0^2+i\epsilon}\right] = (-i\lambda_0)\left[\prod_j\left(\frac{i}{p_j^2-\mu_0^2-\Sigma(p_j^2)+i\epsilon}\right)\right]+\mathcal{O}(\lambda_0^3)\,. \tag{90}$$

Since $\Gamma \sim \mathcal{O}(\lambda_0^2)$**,** $\widetilde{\Gamma}_0 \sim \mathcal{O}(\lambda_0^2)$**, and** $\Sigma \sim \mathcal{O}(\lambda_0)$**, we can also write to this order**

$$\left[\prod_j \frac{i}{p_j^2-\mu_0^2+i\epsilon}\right]\left[3\Gamma(s_0)+\widetilde{\Gamma}(s)+\ldots\right] = \left[\prod_j \frac{i}{p_j^2-\mu_0^2-\Sigma(p_j^2)+i\epsilon}\right]\left[3\Gamma(s_0)+\widetilde{\Gamma}(s)+\ldots\right]+\mathcal{O}(\lambda_0^3) \tag{91}$$

Altogether, we get

$$\begin{aligned} G_0^{(4)}(p_1\ldots p_4) &= \prod_j\left[\frac{i}{p_j^2-\mu_0^2-\Sigma(p_j^2)+i\epsilon}\right]\left[-i\lambda_0+3\Gamma(s_0)+\widetilde{\Gamma}(s)+\widetilde{\Gamma}(t)+\widetilde{\Gamma}(u)\right] \\ &= \left[\prod_j i\Delta(p_j^2)\right]\Gamma_0^{(4)}(p_1\ldots p_4)\,. \end{aligned} \tag{92}$$

Recalling that

$$G_R^{(4)} = Z_\phi^{-2} G_0^{(4)}\,, \quad \Delta_R = Z_\phi^{-1}\Delta\,, \quad \Gamma_R^{(4)} = Z_\phi^2 \Gamma_0^{(4)} \tag{93}$$

we then find from the above equation that

$$\begin{aligned} G_R^{(4)} &= Z_\phi^{-2}\left[Z_\phi^4 \prod_j i\Delta_R(p_j^2)\right] Z_\phi^{-2}\Gamma_R^{(4)} \\ &= \prod_j \left[i\Delta_R(p_j)^2\right] \Gamma_R^{(4)}\,, \end{aligned} \tag{94}$$

which is finite since Δ_R and $\Gamma_R^{(4)}$ are.

Altogether, if we define

$$\phi = Z_\phi^{-1/2}\phi_0\,, \quad \lambda = Z_\lambda^{-1} Z_\phi^2 \lambda_0\,, \quad \mu^2 = \mu_0^2 + \delta\mu^2 (\text{with } \delta\mu^2 = \Sigma(\mu^2)), \tag{95}$$

then the renormalized quantities are finite expressions as functions of μ^2 and λ. This feature, namely that *all divergences, after rewriting everything in terms of λ and μ, can be absorbed*, is the hallmark of a renormalizable theory. And, as we have seen, the S matrix scattering amplitudes are given in terms of the renormalized Green's functions, more precisely the Γ_R for any given process.

BPH Renormalization

BPH=Bogoliubov, Parasink and Hepp (+Zimmerman)

The alternative (but equivalent) approach of BPH renormalization is often more convenient.

First, let us rephrase our result just obtained. We begin once again with

$$\mathcal{L}_0 = \frac{1}{2}[(\partial_\mu \phi_0)(\partial^\mu \phi_0) - \mu_0^2 \phi_0^2] - \frac{\lambda_0}{4!}\phi_0^4 . \tag{96}$$

(here, the 0 subscript refers to before renormalization rather than to just the free particle part of $\mathcal{L}$). If we substitute in terms of the renormalized field and mass and coupling constant, this becomes

$$\begin{aligned}\mathcal{L}_0 &= \frac{1}{2} Z_\phi \left[\partial_\mu \phi \partial^\mu \phi - (\mu^2 - \delta\mu^2)\phi^2\right] - \frac{\left(\frac{Z_\lambda \lambda}{Z_\phi^2}\right)}{4!} Z_\phi^2 \phi^4 \\ &= \frac{1}{2} Z_\phi \left[\partial_\mu \phi \partial^\mu \phi - (\mu^2 - \delta\mu^2)\phi^2\right] - \frac{\lambda Z_\lambda}{4!}\phi^4 . \end{aligned} \tag{97}$$

We rewrite this in the form

$$\mathcal{L}_0 = \mathcal{L} + \Delta\mathcal{L} \tag{98}$$

where

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left[\partial^\mu\phi\partial_\mu\phi - \mu^2\phi^2\right] - \frac{\lambda}{4!}\phi^4 \\ \Delta\mathcal{L} &= (Z_\phi - 1)\frac{1}{2}\left[\partial^\mu\phi\partial_\mu\phi - \mu^2\phi^2\right] + Z_\phi\frac{1}{2}\delta\mu^2\phi^2 - \frac{\lambda(Z_\lambda - 1)}{4!}\phi^4 \end{aligned} \tag{99}$$

where

- $\mathcal{L}$ is the renormalized Lagrangian density
- $\Delta\mathcal{L}$ is the counter term Lagrangian and is explicitly of order $\lambda\mathcal{L}$.

The BPH scheme is the following:

1. Start with $\mathcal{L}$ and compute propagators and vertices.
2. Isolate the divergent 1PI diagrams at order λ (i.e. at one loop).

3. Choose $\Delta\mathcal{L}^{(1)}$ to cancel these divergences.

4. Use $\mathcal{L}^{(1)} = \mathcal{L} + \Delta\mathcal{L}^{(1)}$ to generate 2-loop diagrams.

5. Isolate the infinities of the 2-loop diagrams.

6. Choose $\Delta\mathcal{L}^{(2)}$ (order λ^3) to cancel these infinities.

7. and so forth.

Note that to any given order in λ, this procedure will give a finite expression in terms of the renormalized quantities, λ and μ. In the end, we will have

$$\Delta\mathcal{L} = \Delta\mathcal{L}^{(1)} + \Delta\mathcal{L}^{(2)} + \ldots \tag{100}$$

To analyze further, we need to do some general *power counting*. This will tell us what kinds of counter terms we will need.

A theory is renormalizable if only a small number of counter terms are required and if these counter terms have the same form as terms already in the bare Lagrangian.

To analyze the divergent structure of any Feynman diagram, we introduce the term "superficial degree of divergence", D, which is the number of loop momenta in the numerator minus the number of loop momenta in the denominator. For example, in our fish diagrams of Fig. 1, we find the d^4l in the numerator and $1/(l^2)^2$ (large l) in the denominator, giving $D = 4 - 4 = 0$, implying a logarithmic divergence. In general, D can be computed once we know:

- $B =$ number of external bosons;
- $IB =$ number of internal boson lines;
- $n =$ number of vertices.

The computation of D that we desire is as follows:

1. Since each vertex has 4 lines entering or exiting, and *both* ends of an internal line must terminate on vertices, but only *one* end of an external line must terminate on a vertex, we have

$$4n = 2(IB) + B \tag{101}$$

2. **The number of** *independent* **loop moment that must be integrated over after employing all the δ functions from the vertices is given by**

$$L = IB - n + 1 \tag{102}$$

where the $+1$ is because the 4-momenta delta functions at the n vertices are not all independent given that an overall $\delta(\sum p_i)$ must emerge.

3. **The degree of divergence is**

$$D = 4L - 2(IB) \tag{103}$$

where there is a 4 for each loop d^4l and a -2 for each internal propagator (since they fall off as $1/l^2$).

4. **Using Eq. (102) for L in (103) and then using Eq. (101) to substitute for IB gives**

$$D = 4(IB - n + 1) - 2IB = 2IB - 4n + 4 = 4n - B - 4n + 4 = 4 - B\,. \tag{104}$$

The fact that D does not depend on the internal structure of the diagram is another hallmark of a renormalizable theory. To see how many divergences are possible, we need only look at external structures.

Since $\lambda\phi^4$ has a reflection symmetry under $\phi \rightarrow -\phi$, B must be an even number. Thus, only $B = 2$ and $B = 4$ diagrams are superficially divergent. We reemphasize the fact that this statement is valid to all orders in perturbation theory (i.e. for arbitrarily complicated internal diagram structures).

Of course, we know very well what the most relevant diagrams are (at least at one-loop order). The $B = 2$ diagram at one-loop is the propagator correction which has $D = 2$ and contains quadratically divergent and log divergent pieces. The crucial $B = 4$ diagrams are the 1PI vertex correction diagrams (the fish diagrams at one-loop).

Note however that one can draw diagrams with $D = 0$ or even $D < 0$ that are actually quadratically or logarithmically divergent. An example is the diagram with 6 external bosons connected by a propagator supplemented by a loop correction on any one of the external legs. This is an example of a diagram that superficially has $D = -2$ but is actually quadratically divergent because of the presence of a divergent (in this case, propagator correction) subdiagram. However, this subdiagram will be rendered finite once we have included counterterm Feynman rules that make the basic propagator correction finite. Part of the proof of renormalizability is to show that all $D < 0$ diagrams are rendered finite once counterterms for the basic (1PI) $D = 2$ and $D = 0$ diagrams have been introduced in $\Delta\mathcal{L}$.

Before proceeding to analyze these basic $D = 2$ and $D = 0$ 1PI diagrams and the required counterterms from the BPH perspective, let us first discuss the general case of a ϕ^N theory. We have

$$\begin{aligned} Nn &= (N-4)n + 4n = 2IB + B \\ L &= IB - n + 1 \\ D &= 4L - 2IB \\ &= 4(IB - n + 1) - 2IB \\ &= 2IB - 4n + 4 \\ &= (N-4)n + 4n - B - 4n + 4 \\ &= (4 - B) + (N-4)n \,. \end{aligned} \tag{105}$$

- For $N > 4$, as we move to higher orders, n, we generate higher and higher degrees of divergence. The nature of the divergence is not just a function of the external particle number, but depends also on the order of perturbation theory being considered.

The theory is called *non-renormalizable*.

- For $N < 4$, higher order quickly lead to convergent diagrams. There are only a limited number of actually divergent diagrams.

  For example, for $N = 3$, $D = 4 - B - n$.

  1. For $B = 2$, only diagrams with $n = 1$ or 2 vertices are divergent.
     But, $n = 1$ is topologically impossible for $B = 2$. The smallest possible n is $n = 2$ and the associated diagram (the loop insertion in a line) is log divergent.
     $n = 3$ is topologically impossible.
     $n = 4$ gives two 1PI diagrams that are convergent; of course, the double iteration of the $n = 2$ loop diagram is handled if we deal with the $n = 2$ diagram in the first place.
  2. For $B = 3$, there are no $n = 1$ loop diagrams and all higher n diagrams are convergent.
  3. For $B \geq 4$, any diagram with at least 1 vertex (so that it is a non-trivial diagram) will be convergent.

Thus, there is only *one* divergent diagram and it arises only in the lowest order at which a one-loop diagram propagator correction can be drawn.

The theory is called *super-renormalizable*.

Let us now return to ϕ^4 theory.

We have seen that we should only have to deal with $B = 2$ ($D = 2$) and $B = 4$ ($D = 0$) for renormalization.

For the $D = 2$ case, we have divergences associated with the mass and with the field strength. In a Taylor expansion, we would write

$$\Sigma(p^2) = \Sigma(0) + p^2\Sigma'(0) + \widetilde{\Sigma}(p^2) \tag{106}$$

where $\Sigma(0)$ and $\Sigma'(0)$ are divergent but $\widetilde{\Sigma}(p^2)$ is finite. (It will usually be simpler in this approach to expand about zero-momentum configurations.) This means we need two counterterms:

$$\frac{1}{2}\Sigma(0)\phi^2 + \frac{1}{2}\Sigma'(0)(\partial_\mu\phi)(\partial^\mu\phi) \tag{107}$$

to cancel the divergences. If we introduce these terms into $\Delta\mathcal{L}$ that means we introduce new Feynman rules associated with these Lagrangian structures.

The Feynman rules are easily derived using our standard technique and correspond to point-like insertions onto a single particle line of weights $i\Sigma(0)$ and $i\Sigma'(0)p^2$, respectively. these are the desired signs since $-i\Sigma(p^2)$ is the result of the 1PI loop insertion in a single particle line and so the new $\Delta\mathcal{L}$ terms will have the correct sign to cancel the one-loop terms.

Similarly, for $D = 0, B = 4$ we know that

$$\Gamma^4(p_1, p_2, p_3, p_4) = \Gamma^4(0) + \widetilde{\Gamma}^4(p_1, p_2, p_3, p_4) \tag{108}$$

where we find it convenient to expand about the point where all the $p_{i=1,2,3,4}$ are 0. $\Gamma^4(0)$ we know is log divergent. To cancel this log divergence we introduce a counter term in $\Delta\mathcal{L}$ of the form

$$\Delta\mathcal{L} \ni \frac{i\Gamma^4(0)}{4!}\phi^4 \tag{109}$$

which gives rise to a new 4-point vertex in our set of Feynman rules:

$$-\Gamma^4(0)\,. \tag{110}$$

(To check sign and phase, recall that $\mathcal{L} \ni -\frac{\lambda}{4!}$ produces Feynman rule $-i\lambda$, i.e. multiply by $i4!$.) Thus, (109) will produce Feynman rule (110), which will therefore have the correct sign to cancel the $+\Gamma^4(0)$ coming from the one-loop calculations.

Altogether, we can rewrite $\Delta\mathcal{L}$ in the form

$$\Delta\mathcal{L} = \frac{1}{2}(Z_\phi - 1)[(\partial_\mu\phi)(\partial^\mu\phi) - \mu^2\phi^2] + \frac{1}{2}Z_\phi\delta\mu^2\phi^2 - \frac{\lambda(Z_\lambda - 1)}{4!}\phi^4 \quad (111)$$

if we make the following identifications:

$$\begin{aligned}
\Sigma'(0) &= Z_\phi - 1 \\
\Sigma(0) &= -(Z_\phi - 1)\mu^2 + Z_\phi\delta\mu^2 = -\Sigma'(0)\mu^2 + Z_\phi\delta\mu^2 \\
\Gamma^4(0) &= -i\lambda(1 - Z_\lambda)\,. \qquad (112)
\end{aligned}$$

These results are consistent with our earlier renormalization equations, up to some differences in the treatment of finite terms In particular, our previous "old" approach gave the results:

$$\begin{aligned} Z_\phi &= 1 + \Sigma'(\mu^2) + \mathcal{O}(\lambda_0^2) \\ \mu^2 &= \mu_0^2 + \delta\mu^2 = \mu_0^2 + \Sigma(\mu^2) \\ -iZ_\lambda^{-1}\lambda_0 &= -i\lambda_0 + 3\Gamma(s_0 = 4\mu^2/3)\,. \end{aligned} \tag{113}$$

It is easy to see that (112) is the same as (113), to the order that we are working and as regards the infinite terms. There are finite terms that are treated differently in the two approaches unless it happens that the physical mass $\mu^2 = 0$ so that the subtraction point employed in the BPH scheme and the physical particle mass squared are the same.

For physical mass squared $\mu^2 = 0$, we have the mapping:

$$old\ (at\ \mu^2 = 0) \quad \leftrightarrow \quad new\ (BPH)$$

$$Z_\phi = 1 + \Sigma'(0) \quad \leftrightarrow \quad \Sigma'(0) = Z_\phi - 1 \tag{114}$$

$$\begin{aligned} \delta\mu^2 = \Sigma(0) \quad \leftrightarrow \quad \Sigma(0) &= -\Sigma'(0) \times (\mu^2 = 0) + (1 + \Sigma'(0))\delta\mu^2 \\ &= -0 + \delta\mu^2 + \mathcal{O}(\lambda_0^2) \end{aligned} \tag{115}$$

$$-iZ_\lambda^{-1}\lambda_0 = -i\lambda_0 + 3\Gamma(0) \quad \leftrightarrow \quad \Gamma^4(0) = -i\lambda(1 - Z_\lambda)\,. \tag{116}$$

The equivalences (114) and (115) are trivially obvious. To see the equivalence

(116), let us rewrite the left-hand side as

$$-i\lambda_0 \frac{1 - Z_\lambda}{Z_\lambda} = 3\Gamma(0)\,. \tag{117}$$

Since $1 - Z_\lambda$ is of order λ already, neglecting terms of order λ^2 we may replace Z_λ in the denominator by 1 and λ_0 in the numerator by λ to obtain

$$-i\lambda(1 - Z_\lambda) = 3\Gamma(0)\,. \tag{118}$$

This is the same as the right-hand side of (116) since $\Gamma^4(0) = 3\Gamma(0)$, i.e. the sum of the three fish diagrams evaluated in the $s_0 = t_0 = u_0 = 0$ (off-mass-shell) symmetric configuration.

The fact that the two schemes differ by finite terms if $\mu^2 \neq 0$ is perfectly ok. All that renormalization has to handle is the potentially infinite terms. Schemes that differ as a result of the renormalization quantities Z_ϕ, Z_λ and $\delta\mu^2$ having different finite terms will give the same final answer. However, it is true that some schemes have faster convergence for the perturbation series than others. We shall return to this point later.

It is important to again stress that the divergent graphs ($B = 2$ and $B = 4$) gave divergences that corresponded to 2 and 4 point interactions that were already present in $\mathcal{L}$. We did not have to introduce any new interactions in order to renormalize the theory.

All of this can be extended to higher orders, but this extension is definitely more complex than what we discussed above. One gets involved with divergent subgraphs, "primitively" divergent graphs, Weinberg's theorem, ... We don't have time for this material here. We must turn to the more practical problem at one loop of

Regularization Schemes

A good regularization scheme should maintain Lorentz invariance and the symmetries of the theory.

An old scheme was to employ a Covariant Cutoff.

We will focus on the more modern scheme that is very generally useful, even in the context of supersymmetry, of Dimensional Regularization.

Some key ingredients in this scheme are:

1. Make integrals finite by using $n < 4$ dimensions.

2. Recognize that Feynman integrals are analytic functions of n.

3. Observe that ultraviolet (high-momentum) divergences $\Rightarrow$ poles at $n = 4$.

4. Check that Ward identities are preserved — this is important when discussing gauge theories.

We begin with the example of one of the fish diagrams.

$$\Gamma(p^2) = \frac{(-i\lambda)^2}{2} \int \frac{d^4l}{(2\pi)^4} \frac{i}{(l-p)^2 - \mu^2 + i\epsilon} \frac{i}{l^2 - \mu^2 + i\epsilon} . \qquad (119)$$

To define this integral in n dimensions, take

$$l = (l_0, l_1, \ldots l_{n-1}) , \quad p = (p_0, p_1, p_2, p_3, 0, 0, \ldots) \qquad (120)$$

and write

$$\Gamma(p^2) = \frac{(\lambda)^2}{2} \int \frac{d^nl}{(2\pi)^n} \frac{1}{(l-p)^2 - \mu^2 + i\epsilon} \frac{1}{l^2 - \mu^2 + i\epsilon} , \qquad (121)$$

which is convergent for $n < 4$. We wish to define this for non-integer n. To do so, we first combine the denominators using Feynman parameters and make a Wick rotation.

The Feynman trick is to write

$$\frac{1}{ab} = \int_0^1 \frac{d\alpha}{[\alpha a + (1-\alpha) b]^2} \, . \tag{122}$$

More generally,

$$\frac{1}{a_1 a_2 \ldots a_n} = (n-1)! \int_0^1 \frac{dz_1 \ldots dz_n}{(a_1 z_1 + \ldots a_n z_n)^n} \delta(1 - \sum_{i=1}^{n} z_i) \, . \tag{123}$$

Returning to our $\Gamma(p^2)$, we write

$$\begin{aligned} \frac{1}{(l^2 - \mu^2 + i\epsilon)} \frac{1}{(l-p)^2 - \mu^2 + i\epsilon} &= \int \frac{d\alpha}{[(1-\alpha) l^2 + \alpha (l-p)^2 - \mu^2 + i\epsilon]^2} \\ &= \int \frac{d\alpha}{[(l - \alpha p)^2 + \alpha(1-\alpha) p^2 - \mu^2 + i\epsilon]^2} \\ &\equiv \int \frac{d\alpha}{[k^2 - a^2 + i\epsilon]^2} \, . \end{aligned} \tag{124}$$

The Wick rotation, which is performed on k, takes the Minkowski metric to a Euclidean metric without passing the denominator poles. The poles in question are determined by

$$k^2 - a^2 + i\epsilon = (k^0)^2 - \vec{k}^2 - a^2 + i\epsilon = (k^0)^2 - [(\vec{k}^2 + a^2)^{1/2} - i\epsilon]^2 \quad (125)$$

which gives poles at

$$\begin{aligned} k^0 &= (\vec{k}^2 + a^2)^{1/2} - i\epsilon \\ k^0 &= -(\vec{k}^2 + a^2)^{1/2} + i\epsilon \,. \end{aligned} \quad (126)$$

The idea of the Wick rotation is to rotate the integral along the real k^0 to an integral along the imaginary k^0 axis without wrapping around either of the poles. This specifies a unique direction of rotation. The rotation is pictured in Fig. 3. Since no poles are contained within C, $\oint_C f(k^0) = 0$, and further the contours at ∞ give zero contribution (for well-behaved $f(k^0)$). As a result

$$\int_{real\ axis} f(k^0) dk^0 = \int_{imag.-axis} f(k^0) dk^0 \,, \quad (127)$$

where

$$f(k^0) = \frac{1}{\left[(k^0)^2 - \left\{(\vec{k}^2 + a^2)^{1/2} - i\epsilon\right\}^2\right]^2}. \tag{128}$$

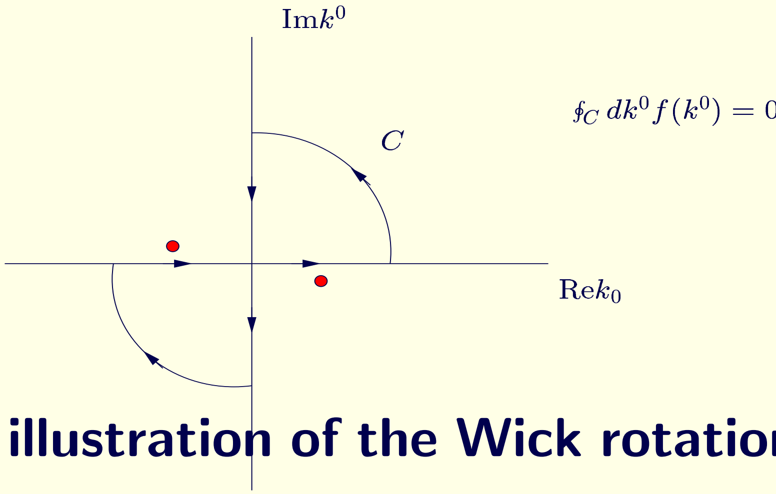


Figure 3: Graphical illustration of the Wick rotation for k^0, with pole locations indicated.

This can be written explicitly as

$$\begin{aligned} \int_{-\infty}^{\infty} dk^0 f(k^0) &= \int_{-i\infty}^{+i\infty} dk^0 f(k^0) \\ &= i \int_{-\infty}^{\infty} d\bar{k}^0 f(i\bar{k}^0) \quad \text{with } k^0 = i\bar{k}^0\,, \end{aligned} \tag{129}$$

and

$$f(i\bar{k}^0) = \frac{1}{\left[(\bar{k}^0)^2 + l_1^2 + l_2^2 + \ldots l_{n-1}^2 + a^2 - i\epsilon\right]^2}. \tag{130}$$

So, now simply relabeling $\bar{k}^0 \to l^0$ in the Euclidean metric form, we obtain

$$\Gamma(p^2) = \frac{i\lambda^2}{2}\int_0^1 d\alpha \int \frac{d^n\vec{l}}{(2\pi)^n}\frac{1}{\left[|\vec{l}|^2 + a^2 - i\epsilon\right]^2}, \vec{l}=\text{Euclidean} \tag{131}$$

where $a^2 = \mu^2 - \alpha(1-\alpha)p^2$. The integral is now independent of angles, which can be integrated out after defining $l = |\vec{l}|$:

$$\begin{aligned}\int d^n\vec{l} &= \int_0^\infty l^{n-1}dl \int_0^{2\pi} d\theta_1 \int_0^\pi \sin\theta_2 d\theta_2 \int_0^\pi \sin^2\theta_3 d\theta_3 \ldots \int_0^\pi \sin^{n-2}\theta_{n-1}d\theta_{n-1} \\ &= \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\int_0^\infty l^{n-1}dl\,. \end{aligned} \tag{132}$$

We employed

$$\int_0^\pi \sin^m(\theta)d\theta = \frac{\sqrt{\pi}\,\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} \tag{133}$$

to obtain the final form above by noting:

$$\begin{aligned} Angle\ stuff &= 2\pi\frac{\sqrt{\pi}\Gamma(1)}{\Gamma(3/2)}\frac{\sqrt{\pi}\Gamma(3/2)}{\Gamma(2)}\cdots\frac{\sqrt{\pi}\Gamma(n-1)}{\Gamma(n/2)} \\ &= 2\pi(\sqrt{\pi})^{n-2}\frac{\Gamma(1)}{\Gamma(n/2)} \\ &= \frac{2\pi^{n/2}}{\Gamma(n/2)}\,. \end{aligned} \tag{134}$$

So, the bottom line is that

$$\Gamma(p^2,n) = \frac{\lambda^2}{2}\frac{2i\pi^{n/2}}{(2\pi)^n\Gamma(n/2)}\int d\alpha\int_0^\infty \frac{l^{n-1}dl}{[l^2+a^2-i\epsilon]^2}\,. \tag{135}$$

An alternative derivation for the angular integrals is to note that

$$\int d\Omega_n = \text{area of unit sphere in } n \text{ dimensions} \equiv S_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}\,, \tag{136}$$

where this latter result might also be derived by noting that $\sqrt{\pi} = \int_{-\infty}^{\infty} dx e^{-x^2}$

so that

$$
\begin{aligned}
(\sqrt{\pi})^n &= \int dx_1 \ldots dx_n e^{-(x_1^2+\ldots x_n^2)} \\
&= \int d\Omega_n \int_0^\infty d|\vec{x}||\vec{x}|^{n-1} e^{-|\vec{x}|^2} \\
&= \int d\Omega_n \frac{1}{2} \int_0^\infty dy y^{\frac{n}{2}-1} e^{-y} \quad y = |\vec{x}|^2 \\
&= \int d\Omega_n \frac{1}{2} \Gamma\left(\frac{n}{2}\right) . \qquad (137)
\end{aligned}
$$

Turning this around we get

$$
\int d\Omega_n \equiv S_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} . \qquad (138)
$$

Now, we must do the final $\int dl$. It is well defined for $0 < \mathrm{Re} n < 4$. But, in fact, we can extend it to any $\mathrm{Re} n < 4$ by multiple parts integration. Thus, only $n \to 4$ needs to be analyzed. First, to perform the $\int dl$ we use the identity:

$$\int_0^\infty \frac{t^{m-1}dt}{(t+a^2)^n} = \frac{1}{(a^2)^{n-m}}\frac{\Gamma(m)\Gamma(n-m)}{\Gamma(n)}\,. \tag{139}$$

Our integral can be rewritten in this form by shifting to $t = l^2$, $dt = 2ldl = 2\sqrt{t}dl$:

$$\begin{aligned}
\int \frac{l^{n-1}dl}{\left[l^2+a^2\right]^2} &= \int \frac{dt}{2\sqrt{t}}\frac{t^{\frac{n-1}{2}}}{\left[t+a^2\right]^2} \\
&= \frac{1}{2}\int \frac{t^{\frac{n}{2}-1}dt}{\left[t+a^2\right]^2} \\
&= \frac{1}{2}\frac{1}{(a^2)^{2-\frac{n}{2}}}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(2-\frac{n}{2}\right)}{\Gamma\left(2\right)} \\
&= \frac{1}{2}\frac{1}{(a^2)^{2-\frac{n}{2}}}\Gamma\left(\frac{n}{2}\right)\Gamma\left(2-\frac{n}{2}\right)\,.
\end{aligned} \tag{140}$$

Now use

$$\Gamma\left(2-\frac{n}{2}\right) = \frac{\Gamma\left(3-\frac{n}{2}\right)}{2-\frac{n}{2}} \stackrel{n\to 4}{\longrightarrow} \frac{2}{4-n} \tag{141}$$

to see that the singularity is a simple pole. To be more precise, let's expand everything around $n = 4$ by writing $n = 4 - 2\epsilon$ (Ryder writes $n = 4 - \epsilon$, so be careful if you are looking at his book.) and use the general result that

$$\Gamma(-k+\epsilon) = \frac{(-1)^k}{k!}\left[\frac{1}{\epsilon} + \psi(k+1) + \frac{1}{2}\epsilon\left\{\frac{\pi^2}{3} + \psi^2(k+1) - \psi'(k+1)\right\} + \mathcal{O}(\epsilon^2)\right] \tag{142}$$

where $\psi(s) \equiv \frac{d\ln\Gamma(s)}{ds}$ and the specific values

$$\begin{aligned} \psi(1) &= -\gamma \\ \psi(k+1) &= 1 + \frac{1}{2} + \ldots + \frac{1}{k} - \gamma \\ \psi'(1) &= \frac{\pi^2}{6} \\ \psi'(k+1) &= \frac{\pi^2}{6} - \sum_{l=1}^{k}\frac{1}{l^2} \end{aligned} \tag{143}$$

will be of particular use. In the immediate context, we write

$$\Gamma\left(2 - \frac{n}{2}\right) = \Gamma\left(2 - \frac{4-2\epsilon}{2}\right)$$

$$= \Gamma(\epsilon)$$

$$= \left[\frac{1}{\epsilon} - \gamma + \frac{1}{2}\epsilon\left\{\frac{\pi^2}{3} + \gamma^2 - \frac{\pi^2}{6}\right\} + \mathcal{O}(\epsilon^2)\right]. \qquad (144)$$

We need to do one more thing. We note that when $n \neq 4$, then λ is not dimensionless any longer. To relate always to a dimensionless coupling constant, we need to rescale with some mass:

$$\lambda_{old} = \lambda_{new}(\mu^2)^{2-\frac{n}{2}=\epsilon} \qquad (145)$$

where λ_{new} is dimensionless. That this is is the appropriate scaling is argued as follows:

1. $\int d^n x(\partial_\mu \phi)^2$ must be dimensionless.

2. Meanwhile $d^n x = d^{4-2\epsilon}x$ has mass dimension in the amount $M^{2\epsilon-4}$, while each ∂_μ has mass dimension M.

3. Thus, the net dimension of $M^{2\epsilon-2}$ must be compensated by the dimension of ϕ^2, implying that ϕ should have dimension $M^{1-\epsilon}$.

4. Next, we examine the $d^n x \lambda \phi^4$ part of the Lagrangian which has mass dimension $M^{2\epsilon-4}\text{dim}[\lambda]M^{4-4\epsilon}$.

To get M^0 requires that λ has mass dimension $M^{2\epsilon}$ which we have written in the conventional form $(\mu^2)^\epsilon$.

Apologies for the multiple use of μ, but both are conventional. The μ above is called the renormalization scale.

However, in what follows, the μ scalar mass will be called m.

Anyway, dropping the new subscript on the λ, our net result is

$$\begin{aligned}\Gamma(p^2) &= \frac{\lambda^2}{2}(\mu^2)^{2\epsilon}\frac{i\pi^{\frac{4-2\epsilon}{2}}}{(2\pi)^{4-2\epsilon}}\left[\frac{1}{\epsilon}-\gamma+\frac{1}{2}\epsilon\left\{\frac{\pi^2}{3}+\gamma^2-\frac{\pi^2}{6}\right\}+\mathcal{O}(\epsilon^2)\right]\int d\alpha\frac{1}{\left[m^2-p^2\alpha(1-\alpha)\right]^\epsilon} \\ &= \frac{\lambda^2}{2}(\mu^2)^\epsilon e^{2\epsilon\ln\mu}\frac{i\pi^2}{(2\pi)^4}e^{\epsilon\ln(4\pi)}\left[\frac{1}{\epsilon}-\gamma+\frac{1}{2}\epsilon\left\{\frac{\pi^2}{3}+\gamma^2-\frac{\pi^2}{6}\right\}\right]\int_0^1 d\alpha e^{-\epsilon\ln[m^2-p^2\alpha(1-\alpha)]},\end{aligned} \tag{146}$$

where we have isolated the power $(\mu^2)^\epsilon$, which is the dimension of λ_{old} which must also be the net dimension of Γ, as we shall see. The above can be rewritten in the form

$$\Gamma(p^2) = \frac{i\pi^2}{2(2\pi)^4}\lambda^2(\mu^2)^\epsilon \left[\frac{1}{\epsilon} - \gamma + \frac{1}{2}\epsilon\left\{\frac{\pi^2}{3} + \gamma^2 - \frac{\pi^2}{6}\right\}\right] \int d\alpha e^{-\epsilon \ln \frac{m^2 - p^2\alpha(1-\alpha)}{4\pi\mu^2}}, \tag{147}$$

which may be expanded up to power ϵ^0 in the form:

$$\frac{i\pi^2}{2(2\pi)^4}\lambda^2(\mu^2)^\epsilon \left[\frac{1}{\epsilon} - \gamma - \int d\alpha \ln \frac{m^2 - p^2\alpha(1-\alpha)}{4\pi\mu^2}\right]. \tag{148}$$

(Notice how the ϵ^1 from the exponential expansion canceled against the $1/\epsilon$ from the ultraviolet singularity. This is a typical thing that one must always be careful to keep.)

The integral is easily evaluated if $p^2 < 0$ (i.e. in the Euclidean domain) and then later continued to $p^2 > 0$. In the Euclidean domain, we can use the result

$$\int d\alpha \ln\left[1 + \frac{4}{a}\alpha(1-\alpha)\right] \stackrel{a\geq 0}{=} -2 + \sqrt{1+a}\ln\left(\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}\right) \tag{149}$$

to obtain

$$\Gamma(p^2) = \frac{i\lambda^2}{32\pi^2}(\mu^2)^\epsilon \left[\frac{1}{\epsilon} - \gamma + 2 + \ln\frac{4\pi\mu^2}{m^2} - \sqrt{1-\frac{4m^2}{p^2}}\ln\left(\frac{\sqrt{1-\frac{4m^2}{p^2}}+1}{\sqrt{1-\frac{4m^2}{p^2}}-1}\right)\right] . \tag{150}$$

This equation will be applied for all three of our fish diagrams, that is for $p^2 = s$, t and u.

Next, we need to turn our attention to Σ. The diagram in question is the one-loop tadpole-like insertion.

$$\begin{aligned} -i\Sigma(p^2) &= \frac{-i\lambda(\mu^2)^\epsilon}{2}\int \frac{d^n l}{(2\pi)^n}\frac{i}{l^2 - m^2 + i\epsilon} \\ &= \frac{\lambda}{2}(\mu^2)^\epsilon \int \frac{d^n l}{(2\pi)^n}\frac{1}{l^2 - m^2 + i\epsilon} \\ &= i\frac{\lambda}{2}(\mu^2)^\epsilon \int \frac{d^n \vec{l}}{(2\pi)^n}\left[\frac{1}{-|\vec{l}|^2 - m^2 + i\epsilon}\right] \\ &= -i\frac{\lambda}{2}(\mu^2)^\epsilon \frac{1}{(2\pi)^n}\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\int_0^\infty \frac{l^{n-1}dl}{l^2+m^2} \end{aligned}$$

$$
\begin{aligned}
&= -i\frac{\lambda}{2}(\mu^2)^\epsilon \frac{1}{(2\pi)^n}\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\frac{1}{2}\int \frac{t^{\frac{n}{2}-1}dt}{t+m^2} \\
&= -i\frac{\lambda}{2}(\mu^2)^\epsilon \frac{1}{(2\pi)^n}\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\frac{1}{2}\frac{1}{(m^2)^{1-\frac{n}{2}}}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(1-\frac{n}{2}\right)}{\Gamma(1)} \\
&= -\frac{\lambda}{2}(\mu^2)^\epsilon \frac{2}{(4\pi)^{n/2}}\frac{1}{2}\frac{1}{(m^2)^{-1+\epsilon}}\Gamma(-1+\epsilon) \\
&= \frac{-i\lambda m^2}{32\pi^2}\left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon \Gamma(-1+\epsilon)\,. \qquad (151)
\end{aligned}
$$

We now do an expansion, using Eq. (142):

$$
\Gamma(-1+\epsilon) = -\left[\frac{1}{\epsilon}+\psi(2)+\mathcal{O}(\epsilon)\right], \qquad (152)
$$

where $\psi(2) = 1-\gamma$. Substituting this in we find:

$$
-i\Sigma(p^2) = \frac{i\lambda m^2}{32\pi^2}\left[\frac{1}{\epsilon}+1-\gamma+\ln\frac{4\pi\mu^2}{m^2}+\mathcal{O}(\epsilon)\right]. \qquad (153)
$$

Again, there was a $\mathcal{O}(\epsilon)$ term from the expansion of

$$\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon} = e^{\epsilon \ln\left(\frac{4\pi\mu^2}{m^2}\right)} \sim 1 + \epsilon \ln\left(\frac{4\pi\mu^2}{m^2}\right) \tag{154}$$

that multiplied the $\frac{1}{\epsilon}$ term to give a term of order $\mathcal{O}(\epsilon^0)$.

Also, we should note: (a) there is no p^2 dependence in this one-loop expression; (b) $\Gamma(-1+\epsilon = 1 - \frac{n}{2})$ also has a pole at $n = 2$ reflecting the quadratic divergence.

We are now in a position to explicitly carry out the renormalization procedure. Define

$$\Delta\mathcal{L} = \frac{1}{2}\Sigma(0)\phi^2 + \frac{1}{2}\Sigma'(0)(\partial_\mu\phi)(\partial^\mu\phi) + \frac{i\Gamma^4(0)}{4!}\phi^4 \tag{155}$$

as before, where $\Sigma'(0)$ is zero because of the lack of p^2 dependence at the present order λ at which we are working. For the non-ϕ^4 part of $\Delta\mathcal{L}$, the appropriate expression is

$$\Delta\mathcal{L} = -\frac{1}{2}\phi^2 \left[\frac{\lambda m^2}{32\pi^2}\right] \left[\frac{1}{\epsilon} + F(\epsilon, \frac{4\pi\mu^2}{m^2})\right], \tag{156}$$

where F is an arbitrary function except for being analytic as $\epsilon \to 0$ and a function of the dimensionless ratio $\frac{\mu^2}{m^2}$. The counter-term Feynman rule associated with this $\Delta\mathcal{L}$ is

$$\frac{-i\lambda m^2}{32\pi^2}\left[\frac{1}{\epsilon}+F\right], \tag{157}$$

so that

$$-i\Sigma(p^2)+C.T. = \frac{i\lambda m^2}{32\pi^2}\left[1-\gamma+\ln\frac{4\pi\mu^2}{m^2}-F\right]. \tag{158}$$

This can now be combined with the 0th order propagator as indicated in the figure to obtain:

$$\begin{aligned}
& bare + [1PI - insertion + CT - insertion] + iterations \\
&= \frac{i}{p^2-m^2}+\frac{i}{p^2-m^2}\frac{i\lambda m^2}{32\pi^2}\left[1-\gamma+\ln\frac{4\pi\mu^2}{m^2}-F\right]\frac{i}{p^2-m^2}+\ldots \\
&= \frac{i}{p^2-m^2+\frac{\lambda m^2}{32\pi^2}\left[1-\gamma+\ln\frac{4\pi\mu^2}{m^2}-F\right]}.
\end{aligned} \tag{159}$$

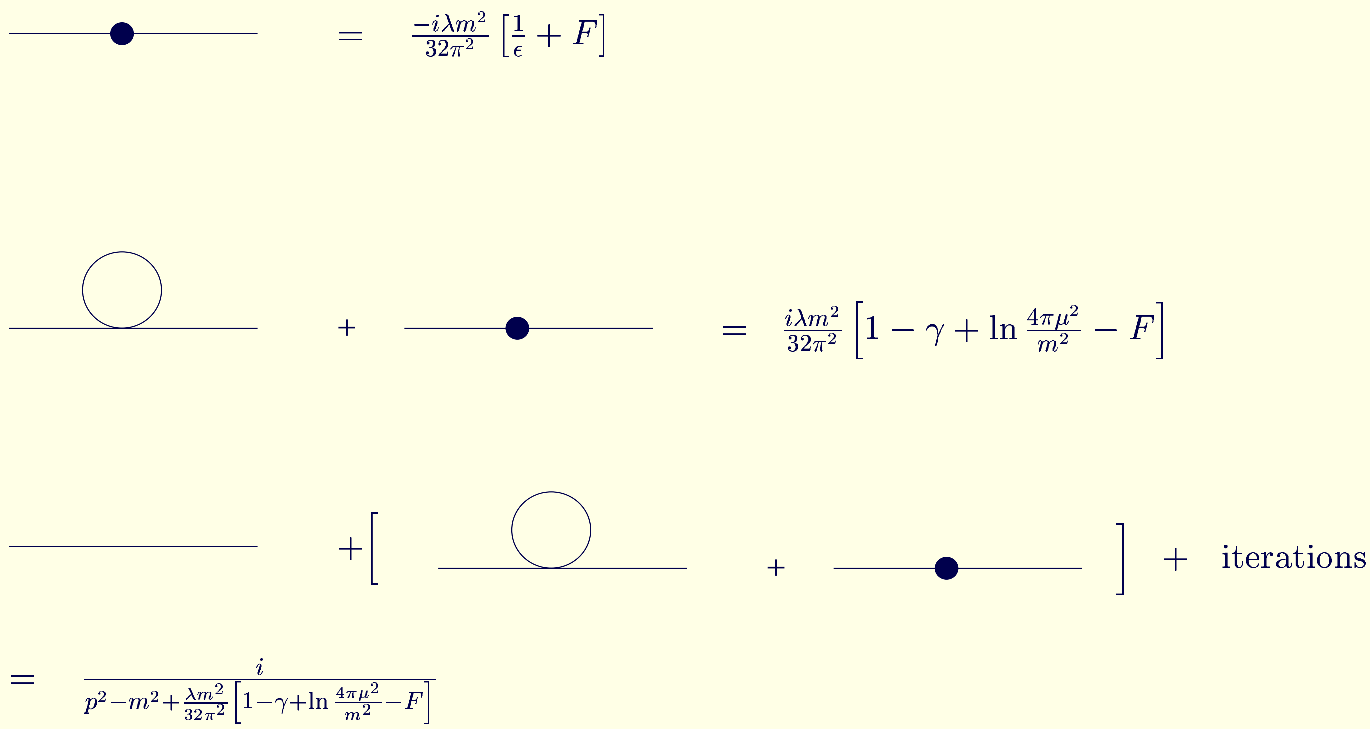


Figure 4: Graphical illustration of counterterm approach for the propagator.

We now need to play the same game for the ϕ^4 term. Recalling Eq. (150), the important singular part is

$$\Gamma(p^2)=\frac{i\lambda^2}{32\pi^2}(\mu^2)^{\epsilon}\frac{1}{\epsilon}\,. \tag{160}$$

and we must remember that since there are 3 fish diagrams this singular structure actually appears with a factor of 3. Thus, we will choose (in

Eq. (155))

$$\Gamma^4(0) = \frac{3i\lambda^2}{32\pi^2}(\mu^2)^\epsilon \left[\frac{1}{\epsilon} + G(\epsilon, \frac{4\pi\mu^2}{m^2})\right] , \qquad (161)$$

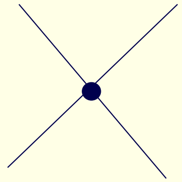
for which the Feynman rule deriving from the $\frac{i\Gamma^4(0)}{4!}\phi^4$ interaction is given by

$$-\Gamma^4(0) = -\frac{3i\lambda^2}{32\pi^2}(\mu^2)^\epsilon \left[\frac{1}{\epsilon} + G(\epsilon, \frac{4\pi\mu^2}{m^2})\right] \qquad (162)$$

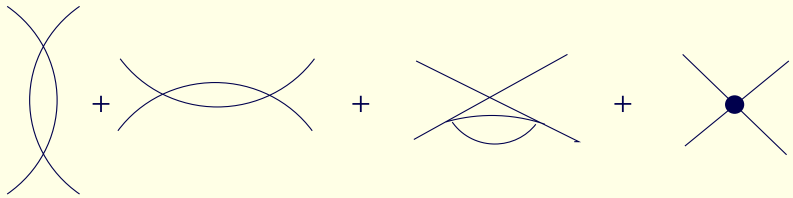
so that the sum of the 3 fish diagrams plus the vertex counter diagram,

$$\begin{aligned} &\Gamma(s) + \Gamma(t) + \Gamma(u) - \Gamma^4(0) \\ = \quad & 3\frac{i\lambda^2}{32\pi^2}(\mu^2)^\epsilon \left[-\gamma + 2 + \ln\frac{4\pi\mu^2}{m^2} - \frac{1}{3}\sum_{z=s,t,u}\sqrt{1-\frac{4m^2}{z}}\ln\left(\frac{\sqrt{1-\frac{4m^2}{z}}+1}{\sqrt{1-\frac{4m^2}{z}}-1}\right) - G(\epsilon, \frac{4\pi\mu^2}{m^2})\right] , \end{aligned} \qquad (163)$$

is finite.



$$= \quad -\frac{3i\lambda^2}{32\pi^2}(\mu^2)^\epsilon\left[\frac{1}{\epsilon}+G(\epsilon,\frac{4\pi\mu^2}{m^2})\right]$$



$$= \quad 3\frac{i\lambda^2}{32\pi^2}(\mu^2)^\epsilon\left[-\gamma+2+\ln\frac{4\pi\mu^2}{m^2}-\frac{1}{3}\sum_{z=s,t,u}\sqrt{1-\frac{4m^2}{z}}\ln\left(\frac{\sqrt{1-\frac{4m^2}{z}}+1}{\sqrt{1-\frac{4m^2}{z}}-1}\right)-G(\epsilon,\frac{4\pi\mu^2}{m^2})\right]$$

Figure 5: Graphical illustration of counterterm approach for vertex.

Subtraction "Schemes"

1. **In the so-called** *minimal subtraction* **(MS) scheme, we take $F = G = 0$.**

2. **In the $\overline{\mathrm{MS}}$ subtraction scheme, we choose**

$$F = G = -\gamma + \ln 4\pi \tag{164}$$

so as to absorb the obvious constants that appear in both calculations.

As it happens, the $\overline{\mathrm{MS}}$ scheme often leads to smaller coefficients for higher order correction expansions, but for the moment we will stick to the MS scheme.

In the MS scheme, we have the following identifications with the general form of (after taking $\mu \to m$ in the basic form to avoid confusion with the renormalization scale μ and going to the dimensionless coupling λ in n dimensions)

$$\Delta\mathcal{L} = \frac{1}{2}(Z_\phi - 1)[(\partial_\mu\phi)(\partial^\mu\phi) - m^2\phi^2] + \frac{1}{2}Z_\phi \delta m^2 \phi^2 - \frac{\lambda(\mu^2)^\epsilon(Z_\lambda - 1)}{4!}\phi^4 . \tag{165}$$

First, at this order only,

$$\frac{1}{2}(Z_\phi - 1)(\partial_\mu\phi)(\partial^\mu\phi) = 0 \tag{166}$$

since $\Sigma'(0) = 0$. The two non-trivial identifications are:

$$\frac{1}{2}Z_\phi \delta m^2 \phi^2 = -\frac{1}{2}\phi^2 \frac{\lambda m^2}{32\pi^2}\frac{1}{\epsilon} . \tag{167}$$

$$-\frac{\lambda(\mu^2)^\epsilon}{4!}(Z_\lambda - 1)\phi^4 = -\frac{\lambda(\mu^2)^\epsilon}{4!}\phi^4 \frac{3\lambda}{32\pi^2}\frac{1}{\epsilon}\,. \tag{168}$$

Reading off this matching, we find

$$\begin{aligned} Z_\phi &= 1 \\ \delta m^2 &= -\frac{\lambda m^2}{32\pi^2}\frac{1}{\epsilon} \\ (Z_\lambda - 1) &= \frac{3\lambda}{32\pi^2}\frac{1}{\epsilon}\,. \end{aligned} \tag{169}$$

What's in the choice of a subtraction scheme?

As we have seen, there is ambiguity in the choice of the finite part of the counterterms. This ambiguity leads to different possible definitions of the coupling constant employed in the perturbative expansions. It turns out to be the case that some coupling constant definitions lead to a systematically better expansion in that, for example, using QED notation,

$$prediction = e^2 + ae^4 \tag{170}$$

where a may be small for one definition of e but big for another definition. It

turns out that this is particularly relevant for QCD.

To see what is at issue more clearly, let us imagine a theory of Quantum Imaginary Dynamics, QID. Assume that in the far distant future someone is able solve QID exactly so that predictions for important measurements are precisely related to one another.

- At the present time, we imagine that there are two high precision measurements that are made:
  1. the "QID" Josephson Effect"

$$J_I = 0.200000000\ (13) \tag{171}$$

  2. the magnetic moment of the QID electron

$$a_e^I = 0.33333333(12)\,. \tag{172}$$

- The exact solution of QID in terms of a parameter x (to be measured) yields the results

$$\begin{aligned} J_I &= x \\ a_e^I &= \frac{x}{1-2x}\,. \end{aligned} \tag{173}$$

Obviously, if x is measured by J_I (so that $x = 0.2\ldots$) then this exact solution predicts $a_e^I = 0.333\ldots$.

- Unfortunately for the inhabitants of QID land, their stupid physicists have not solved QID exactly and must instead resort to QID perturbation theory.

  They find it convenient to expand in terms of a parameter y which appears naturally in the calculation of QID perturbation theory diagrams. It happens that

$$x = y(1 + 10y) \tag{174}$$

  but the physicists are unaware of that. They attach no particular significance to x.

- Now, expanding we have

$$\begin{aligned} J_I &= y + 10y^2 \\ a_e^I &= y + 12y^2 + 44y^3 + \ldots . \end{aligned} \tag{175}$$

Keeping only the leading "tree-level" terms linear in y, a measurement of $J_I = 0.2 = y$ *predicts* $a_e^I = 0.2$, in bad disagreement with experiment. This was all the professors could manage.

- After several years of lengthy QID computations, the graduate students obtained the next order, i.e. $\mathcal{O}(y^2)$, results so that now they had

$$y + 10y^2 = 0.2\,, \quad \Rightarrow \quad y = 0.1 \tag{176}$$

yielding the prediction $a_e^I = y + 12y^2 = 0.22$, which still gave bad agreement, despite the fact that $y = 0.1$ seems like a small expansion parameter. The problem is that the coefficients in the expansion of a_e^I in terms of y are very big.

- Finally, some clever physicist says lets choose an expansion parameter such that the coefficients in the expansion are better behaved, and perhaps the expansion parameter has additional physical motivation as well.

An obvious guess, once you compute to order y^2 and realize that $x = y + 10y^2$ has a big y^2 coefficient, is to use x instead of y, so that

$J_I = x + \mathcal{O}(x^3)$. Then, you find from your a_e^I calculation that

$$a_e^I = x + 2x^2 + \mathcal{O}(x^3) \quad (177)$$

which is also clearly better behaved in that the expansion coefficient is much smaller than before when y was used. Using x, one concludes that the measurement $J_I = 0.2 \Rightarrow x = 0.2$ which, in turn, $\Rightarrow a_e^I \sim 0.28$ (up to corrections of order x^3 which they hope are small).

Well this is much better, but still not exactly perfect.

Note that even though $x = 0.2 > y = 0.1$, the series is much more convergent in x.

- After another 3 years another graduate student manages to compute the 500+ Feynman diagrams (using some advanced algebraic manipulation programs and automated integration routines) needed to obtain the next terms at $\mathcal{O}(x^3)$ for J_I and a_e^I, discovering that $J_I = x + \mathcal{O}(x^4)$ and that $a_e^I = x + 2x^2 + 4x^3 + \mathcal{O}(x^4)$.

  The measurement of J_I still gives $x = 0.2$ and, to the computed order, $a_e^I = 0.312$. At this point, the measured value of $a_e^I = 0.333$ is in sufficiently good agreement with the prediction based on the $J_I = 0.2$ measurement and the perturbative series that QID is declared a correct theory. (Although some uncertainty must inevitably remain since the $\mathcal{O}(x^4)$ terms would require computing several thousand highly complex Feynman diagrams and the computation is estimated to take more than 5 years — a project that is deemed excessive even for assigning to a graduate student.)

- In QED, we are fortunate to have a physical observation (the charge of the electron) which we use to define our expansion parameter e and we are lucky in that not only is e small (more precisely, $\alpha = \frac{e^2}{4\pi} \ll 1$), but also the coefficients of powers of α that appear in calculating other quantities in terms of α are small.

- In QCD, we have no such direct measurement of g_s and we must be clever in the way in which we define our g_s (by trying to make it as physical as possible) in order to keep our perturbation theory coefficients in the expansion in powers of $\alpha_s = \frac{g_s^2}{4\pi}$ under control.

  Experience has shown that the $\overline{\mathrm{MS}}$ scheme for defining the renormalized coupling constant is much better in this sense than the MS scheme.

Renormalization Group: "Moving" Coupling Constant and Mass in ϕ^4 theory

The understanding of how to use the renormalization process we have just gone through to make predictions for how the coupling λ (as measured by a four-point interaction or scattering) and the mass m (as "measured" by effective strength in probes of mass, e.g. through Yukawa interactions) change as a function of the energy scale of the probe, is of fundamental importance.

To proceed, let us summarize the results so far, and generalize a bit.

We recall that

$$\begin{aligned} \phi &= Z_\phi^{-1/2}\phi_0 \\ \lambda(\mu^2)^\epsilon &= Z_\lambda^{-1} Z_\phi^2 \lambda_0 \\ m^2 &= m_0^2 + \delta m^2 \,. \end{aligned} \tag{178}$$

Our calculation so far has given

$$\lambda_0 = (\mu^2)^\epsilon \lambda Z_\lambda = (\mu^2)^\epsilon \lambda \left[1 + \frac{3\lambda}{32\pi^2}\frac{1}{\epsilon} \right] \tag{179}$$

$$m_0^2 \;=\; m^2 - \delta m^2 = m^2 \left[1 + \frac{\lambda}{32\pi^2}\frac{1}{\epsilon}\right] . \tag{180}$$

In higher orders this and the analogues for the other cases will take the forms

$$\begin{aligned}
\lambda_0 &= (\mu^2)^\epsilon \left[\lambda + \sum_{k=1}^{\infty} \frac{a_k(\lambda)}{\epsilon^k}\right] \\
m_0^2 &= m^2 \left[1 + \sum_{k=1}^{\infty} \frac{b_k(\lambda)}{\epsilon^k}\right] \\
Z_\phi &= \left[1 + \sum_{k=1}^{\infty} \frac{c_k(\lambda)}{\epsilon^k}\right] .
\end{aligned} \tag{181}$$

So, let us take $\mu\frac{\partial}{\partial\mu}$ of one of the bare parameters, for example λ_0. Well, obviously λ_0, the input Lagrangian parameter, knows nothing about the arbitrary scale μ that we introduced as part of our renormalization process in extra dimensions. However, the renormalized parameters λ and m are implicit functions of the renormalization scale μ: $\lambda(\mu)$, $m(\mu)$. Thus, working to one-loop order we have

$$\mu\frac{\partial\lambda_0}{\partial\mu} = 0$$
$$= 2\epsilon\lambda(\mu^2)^\epsilon\left[1+\frac{3\lambda}{32\pi^2\epsilon}\right]+(\mu^2)^\epsilon\left[\mu\frac{\partial\lambda}{\partial\mu}+\frac{6\lambda}{32\pi^2\epsilon}\mu\frac{\partial\lambda}{\partial\mu}\right]. \quad (182)$$

Solving, we find

$$\mu\frac{\partial\lambda}{\partial\mu}=\frac{-2\epsilon\lambda(\mu^2)^\epsilon\left[1+\frac{3\lambda}{32\pi^2\epsilon}\right]}{(\mu^2)^\epsilon\left[1+\frac{6\lambda}{32\pi^2\epsilon}\right]}. \quad (183)$$

Now, we want to take the $\epsilon \to 0$ limit. In so doing, we must stick very consistently to a definite order in perturbation theory, writing

$$\mu\frac{\partial\lambda}{\partial\mu} = -2\epsilon\lambda\left[1+\frac{3\lambda}{32\pi^2\epsilon}-\frac{6\lambda}{32\pi^2\epsilon}+\mathcal{O}(\lambda^2)\right] \quad (184)$$
$$\stackrel{\epsilon\to 0}{=} \frac{3\lambda^2}{16\pi^2}$$
$$\equiv \beta(\lambda). \quad (185)$$

(A more careful treatment is possible, and perhaps we shall get to it.)

Similarly, we have

$$\mu\frac{\partial m_0^2}{\partial\mu} = 0$$
$$= \mu\frac{\partial m^2}{\partial\mu}\left[1+\frac{\lambda}{32\pi^2\epsilon}\right] + m^2\mu\frac{\partial\lambda}{\partial\mu}\frac{1}{32\pi^2}\frac{1}{\epsilon}, \qquad (186)$$

which can be solved to give

$$\mu\frac{\partial m^2}{\partial\mu} = -m^2\mu\frac{\partial\lambda}{\partial\mu}\frac{1}{32\pi^2}\frac{1}{\epsilon}\left[1-\frac{\lambda}{32\pi^2}\frac{1}{\epsilon}\right], \qquad (187)$$

into which we substitute the result of Eq. (184) (i.e. $\mu\frac{\partial\lambda}{\partial\mu} = -2\epsilon\lambda\left[1-\frac{3\lambda}{32\pi^2}\frac{1}{\epsilon}\right]$) to obtain

$$\mu\frac{\partial m^2}{\partial\mu} = -m^2(-2\epsilon\lambda)\left[1-\frac{3\lambda}{32\pi^2}\frac{1}{\epsilon}\right]\frac{1}{32\pi^2}\frac{1}{\epsilon}$$
$$\overset{\epsilon\to 0}{=} m^2\left[\frac{\lambda}{16\pi^2}+\mathcal{O}(\lambda^2)\right]. \qquad (188)$$

This, we usually write in the form

$$\frac{1}{2}\mu\frac{\partial \ln m^2}{\partial \mu} = \frac{\lambda}{32\pi^2} \equiv \gamma_m(\lambda(\mu)) \,. \tag{189}$$

So, now the question is how to interpret these $\mu\frac{\partial}{\partial\mu}$ derivative forms. We will give a more precise way of phrasing this interpretation in terms of the renormalization group equations (RGEs), but let us first attempt a naive/intuitive explanation. For this, let us return to the expression for

$$\begin{aligned} & \text{bare vertex} + \sum \text{fish diagrams} + \text{vertex counter term} \\ = \;& -i\lambda(\mu)(\mu^2)^\epsilon + i\frac{\lambda^2}{32\pi^2}(\mu^2)^\epsilon 3\left[-\gamma + 2 + \ln\frac{4\pi\mu^2}{m^2} - \frac{1}{3}A(s,t,u)\right], \end{aligned} \tag{190}$$

where

$$A(s,t,u) = \sum_{z=s,t,u} \beta(z)\ln\frac{\beta(z)+1}{\beta(z)-1}, \quad \text{with} \quad \beta(z) = \sqrt{1-\frac{4m^2}{z}} \,. \tag{191}$$

Let us imagine first performing a measurement of the four-point interaction at an energy such that $s, t, u \sim s_0$. Then, if $\mu^2 \sim s_0$ and $m^2(\mu) \sim s_0$ also, everything in the above expression is simply set by the size of $\lambda(\mu \sim \sqrt{s_0})$. There are no big correction factors coming from the logarithms in the expression.

Next, we imagine increasing the energy at which the experiment operates so that $s, t, u \to f s_0$. Let us examine the behavior of $A(s,t,u)$ in this limits. First, we note that $\beta(s) \sim 1 - \frac{2m^2}{f s_0}$ for large f so that

$$A(s,t,u) \to 3 \ln \frac{2}{\left(\frac{-2m^2}{f s_0}\right)} \sim 3 \ln f + finite\,. \tag{192}$$

The idea is to cancel this large logarithmic stuff with some appropriate change of μ that depends on f. We need

$$\ln \mu^2(f) - \frac{1}{3} A(s,t,u) \sim \ln \mu^2(f) - \ln f + finite \sim finite\,. \tag{193}$$

The obvious way to accomplish this is if

$$\mu^2(f) \sim f s_0\,. \tag{194}$$

If we do not make this change in μ as a function of f, then $\lambda(\mu)$ (the leading term) would not be a very good measure of the strength of the 4-point interaction in the high energy experiment. If we *do* scale up μ^2 by a factor of f, then there are no big logarithmic factors multiplying the $\lambda^2(\mu)$ correction to the leading $\lambda(\mu)$ term in Eq (190). As a result, the leading $\lambda(\mu)$ component of Eq. (190) is a proper measure of the strength of the interaction at this higher energy scale if $\mu^2 \sim f s_0$ since the $\lambda^2(\mu) \times$ modest size stuff is a small correction. Thus, we will wish to know how $\lambda(\mu)$ behaves as μ^2 is increased by some factor f. This is clearly determined by the functional form of $\beta(\lambda)$ through the differential equation given in Eq. (185).

In short, to avoid ratios of large scales in the propagator and vertex after renormalization, we should take μ of order the energy scale E of the experimental probe. Then, very roughly,

$$4 - point\ scattering\ amplitude|_E \sim -i\lambda(\mu = E)\,, \tag{195}$$

so that the behavior of $\lambda(\mu)$ as a function of μ reveals the physical behavior of 4-point scattering as a function of E.

So, now let us solve the differential equation

$$\mu \frac{\partial \lambda(\mu)}{\partial \mu} = \frac{3\lambda^2(\mu)}{16\pi^2} \tag{196}$$

by writing it in the form

$$\frac{d\lambda}{\lambda^2} = \frac{d\mu}{\mu} \frac{3}{16\pi^2} \tag{197}$$

which integrates to

$$-\frac{1}{\lambda} = \frac{3}{16\pi^2} \ln \mu + c \tag{198}$$

where c can be fixed by specifying λ at some low μ scale, say μ_s:

$$-\frac{1}{\lambda_s} = \frac{3}{16\pi^2} \ln \mu_s + c\,. \tag{199}$$

Subtracting, we obtain

$$\frac{1}{\lambda} - \frac{1}{\lambda_s} = \frac{3}{16\pi^2} \ln \frac{\mu_s}{\mu} \tag{200}$$

which can be rewritten in the form

$$\boldsymbol{\lambda} = \frac{\boldsymbol{\lambda_s}}{1 - \frac{3}{16\pi^2}\boldsymbol{\lambda_s} \ln \frac{\mu}{\mu_s}}, \tag{201}$$

where, to repeat, $\lambda_s = \lambda(\mu = \mu_s)$.

Note that λ increases with increasing μ if we start with a small λ_s. At larger μ (shorter distance) this means that the number of orders of λ needed for a reliable result must increase. Eventually, we would leave the domain where λ is sufficiently small that a perturbative result is trustworthy, regardless of how many orders in λ are kept.

Conversely, in $\lambda\phi^4$ theory, perturbation theory becomes more reliable at larger distances (small μ). Further, since λ is small at large distances, the perturbative approach to defining asymptotic states should also be reliable.

However, if the sign of $\mu\frac{\partial\lambda}{\partial\mu}$ were to be negative, then quite the reverse situation arises:

1. short distance behavior will be perturbative;

2. long distance behavior would not be easily computed, and asymptotic states states might not be easily defined.

A negative β function is characteristic of non-Abelian groups, most especially QCD based on SU(3).

Anyway, now let's turn to the evolution of m^2. We had

$$\frac{1}{2}\mu\frac{\partial \ln m^2}{\partial\mu} = \frac{\lambda(\mu)}{32\pi^2} \tag{202}$$

which implies

$$\ln m^2 = \int \frac{d\mu}{\mu}\frac{\lambda(\mu)}{16\pi^2}\,. \tag{203}$$

To integrate this easily, we need to work a bit on our $\lambda(\mu)$ solution:

$$\begin{aligned} \lambda &= \frac{\lambda_s}{1-\frac{3}{16\pi^2}\lambda_s \ln\frac{\mu}{\mu_s}} \\ &= \frac{1}{\frac{1}{\lambda_s}-\frac{3}{16\pi^2}\ln\frac{\mu}{\mu_s}} \\ &= \frac{-1}{\frac{3}{16\pi^2}\ln\frac{\mu}{\Lambda}} \end{aligned} \tag{204}$$

provided we define

$$\frac{1}{\lambda_s} \equiv \frac{3}{16\pi^2} \ln \frac{\Lambda}{\mu_s} . \tag{205}$$

For ϕ^4 theory, it is convenient to think of using this expression for $\mu < \Lambda$ so that λ as written above will be positive (as needed for stability against large values of ϕ). We then obtain

$$\begin{aligned} \ln m^2 &= -\frac{1}{3} \int \frac{d \ln \frac{\mu}{\Lambda}}{\ln \frac{\mu}{\Lambda}} \\ &= -\frac{1}{3} \ln \left(\ln \frac{\mu}{\Lambda} \right) + c , \end{aligned} \tag{206}$$

implying that

$$m^2 = e^c \left(\ln \frac{\mu}{\Lambda} \right)^{-1/3} \equiv \overline{c} \left[-\lambda(\mu) \right]^{1/3} . \tag{207}$$

Usually the integration constant is determined by the requirement that

$$m^2(m^2_{physical}) = m^2_{physical} \tag{208}$$

where $m^2_{physical}$ is the physical pole location. (This is not possible in QCD, but it is OK in the Higgs sector.) Using this condition gives

$$\overline{c}[-\lambda(m_{physical})]^{1/3} = m^2_{physical}\,, \quad \Rightarrow \quad \overline{c} = m^2_{physical}[-\lambda(m_{physical})]^{-1/3}\,, \tag{209}$$

implying that

$$m^2(\mu) = m^2_{physical}\left[\frac{\lambda(\mu)}{\lambda(m_{physical})}\right]^{1/3}\,. \tag{210}$$

In the present case of ϕ^4 theory, if $\mu\uparrow$ then $\lambda(\mu)\uparrow$ and, hence, $m^2(\mu)\uparrow$. The opposite happens in QCD.

The meaning of this type of $m^2(\mu)$ dependence will only become apparent when we discuss the RGE approach to all this. Basically, while the μ dependence of λ is primarily set by the need to cancel large logarithms, the μ dependence of m^2 is set up to cancel growth of subdominant $\ln\ln$ terms.

We can also examine what happens in the case of Z_ϕ. From the power series definition given earlier in Eq. (181) and a careful treatment of the ϵ expansions, we find

$$\mu\frac{\partial \ln Z_\phi}{\partial\mu} = -2\lambda\frac{dc_1}{d\lambda}\,. \tag{211}$$

Defining

$$\gamma_d = \frac{1}{2}\mu\frac{\partial \ln Z_\phi}{\partial \mu} = \frac{1}{2}\frac{\mu\frac{\partial Z_\phi}{\partial \mu}}{Z_\phi} \tag{212}$$

yields

$$\gamma_d(\lambda) = -\lambda\frac{dc_1}{d\lambda}. \tag{213}$$

In ϕ^4 theory, the first non-zero term for c_1 is at order λ^2 (from the 3-finger propagator insertion graph) and one finds

$$\gamma_d(\lambda) = \frac{1}{12}\left(\frac{\lambda}{16\pi^2}\right)^2 + \mathcal{O}(\lambda^3). \tag{214}$$

γ_d is called the "anomalous dimension" for reasons that will become apparent when we turn to the RGEs.

Incidentally, you will have noticed that $\frac{\lambda}{16\pi^2}$ appears as a factor in all of our one-loop type contributions. It is this that is the real expansion parameter. Obviously, the typical one-loop integration factor $\frac{1}{16\pi^2}$ greatly helps the convergence of the perturbation series.

Renormalization Group Equations for ϕ^4.

The renormalization group is a group of transformations in which $\mu \to e^t \mu$. For our purposes, it is characterized by the $\mu \frac{\partial}{\partial \mu}$ type derivatives.

Let us examine a general Green's function (after amputating the external legs):

$$\Gamma_0^N(p_1 \ldots p_N; \lambda_0, m_0, \epsilon) = Z_\phi^{-N/2} \Gamma^N(p_1 \ldots p_N; \lambda(\mu), m(\mu), \mu, \epsilon) \quad (215)$$

where Γ^N is the renormalized amputated Green's function and is finite as $\epsilon \to 0$. The Γ_0 is independent of μ so that, using the chain rule,

$$\begin{aligned} \mu \frac{\partial \Gamma_0^N}{\partial \mu} &= 0 \\ &= \left[\mu \frac{\partial}{\partial \mu} + \mu \frac{\partial \lambda}{\partial \mu} \frac{\partial}{\partial \lambda} + \mu \frac{\partial m}{\partial \mu} \frac{\partial}{\partial m} - \frac{N}{2} \frac{\mu \frac{\partial Z_\phi}{\partial \mu}}{Z_\phi} \right] \Gamma^N \\ &= \left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + m \gamma_m(\lambda) \frac{\partial}{\partial m} - N \gamma_d(\lambda) \right] \Gamma^N . \quad (216) \end{aligned}$$

This is the renormalization group equation for Γ^N.

We can derive another equation obeyed by Γ^N by noting that it has an "engineering" dimension equal to

$$4 - N + \epsilon(N-2)\,. \tag{217}$$

This is most easily seen by noting that the effective term induced in $\mathcal{L}$ by Γ^N is $\propto \Gamma^N \phi^N$ and we know that

$$\int d^n x \Gamma^N \phi^N = dimensionless \tag{218}$$

while ϕ has dimensions $M^{\frac{n}{2}-1}$ as argued earlier. So, if Γ^N has dimension $M^{n-N(\frac{n}{2}-1)}$ then

$$\int d^n x \Gamma^N \phi^N \propto M^{-n} M^{n-N(\frac{n}{2}-1)} M^{N(\frac{n}{2}-1)} \propto M^0\,. \tag{219}$$

Substituting $n = 4 - 2\epsilon$ gives

$$M^{n-N(\frac{n}{2}-1)} = M^{4-2\epsilon-N(1-\epsilon)} = M^{4-N+\epsilon(N-2)}\,. \tag{220}$$

This dimension can only be carried by the dimensionful parameters of Γ^N. Hence, we must have

$$\left[\mu\frac{\partial}{\partial\mu} + s\frac{\partial}{\partial s} + m\frac{\partial}{\partial m} - \{4 - N + \epsilon(N-2)\}\right]\Gamma^N(sp_1 \ldots sp_N; m, \lambda, \mu, \epsilon) = 0\,, \tag{221}$$

where we have used s to set the scale of the momenta. In writing the above, we have used the fact that Γ^N will be a sum of terms, each of which is a product of various powers of our dimension-carrying objects and that these powers are "probed" by the logarithmic $\mu\frac{\partial}{\partial\mu}$ derivative. For example,

$$\begin{aligned} \mu\frac{\partial}{\partial\mu}\mu^p &= p\mu^p \\ m\frac{\partial}{\partial m}m^q &= qm^q \\ s\frac{\partial}{\partial s}(sp_1\cdot sp_2)^r &= s\frac{\partial}{\partial s}s^{2r}(p_1\cdot p_2)^r = 2rs^{2r}(p_1\cdot p_2)^r = 2r(sp_1\cdot sp_2)^r \end{aligned} \tag{222}$$

and similarly for other invariants formed from momenta.

We use this 2nd equation to eliminate $\mu\frac{\partial}{\partial\mu}$ from Eq. (216) to obtain in the limit of $\epsilon \longrightarrow 0$:

$$\left[-s\frac{\partial}{\partial s} - m\frac{\partial}{\partial m} + (4-N) + \beta(\lambda)\frac{\partial}{\partial \lambda} + \gamma_m(\lambda)m\frac{\partial}{\partial m} - N\gamma_d(\lambda)\right]\Gamma^N(sp; m, \lambda, \mu) = 0\,, \tag{223}$$

or, using the notation $4 - N = d_N$, the "naive" engineering dimension in 4 dimensions,

$$\left[-s\frac{\partial}{\partial s} + \beta(\lambda)\frac{\partial}{\partial \lambda} + (\gamma_m(\lambda)-1)m\frac{\partial}{\partial m} + d_N - N\gamma_d(\lambda)\right]\Gamma^N(sp; m, \lambda, \mu) = 0\,. \tag{224}$$

(At all stages above, one should keep in mind that $\lambda = \lambda(\mu)$ and $m = m(\mu)$.) Eq. (224) expresses the fact that a change in s may be compensated by a change in m and λ and an overall engineering factor. So we expect the solution to be of the form:

$$\Gamma^N(sp; m, \lambda, \mu) = f(s)\Gamma^N(p; m(s), \lambda(s), \mu) \tag{225}$$

in which case

$$\frac{\partial}{\partial s}\Gamma^N(sp; m, \lambda, \mu) = \frac{\partial f(s)}{\partial s}\Gamma^N(p; m(s), \lambda(s), \mu) + f(s)\left[\frac{\partial m(s)}{\partial s}\frac{\partial \Gamma^N(p; \ldots)}{\partial m} + \frac{\partial \lambda(s)}{\partial s}\frac{\partial \Gamma^N(p; \ldots)}{\partial \lambda}\right] \tag{226}$$

or

$$s\frac{\partial}{\partial s}\Gamma^N(sp; m, \lambda, \mu) \quad = \quad \left[s\frac{\partial f}{\partial s} + f(s)s\frac{\partial m(s)}{\partial s}\frac{\partial}{\partial m} + f(s)s\frac{\partial \lambda(s)}{\partial s}\frac{\partial}{\partial \lambda}\right]\Gamma^N(p; m(s), \lambda(s), \mu)$$

$$= \left[s\frac{\partial f}{\partial s} + f(s)s\frac{\partial m(s)}{\partial s}\frac{\partial}{\partial m} + f(s)s\frac{\partial \lambda(s)}{\partial s}\frac{\partial}{\partial \lambda}\right]\frac{1}{f(s)}\Gamma^N(sp; m, \lambda, \mu) \quad (227)$$

which we choose to rewrite in the form:

$$\left[-s\frac{\partial}{\partial s} + \frac{s}{f(s)}\frac{\partial f(s)}{\partial s} + s\frac{\partial m(s)}{\partial s}\frac{\partial}{\partial m} + s\frac{\partial \lambda(s)}{\partial s}\frac{\partial}{\partial \lambda}\right]\Gamma^N(sp; m, \lambda, \mu) = 0 \quad (228)$$

We now compare Eq. (224) to Eq. (228). Matching coefficients, we conclude

$$s\frac{\partial \lambda(s)}{\partial s} = \beta(\lambda(s)) \quad (229)$$

$$s\frac{\partial m(s)}{\partial s} = m[\gamma_m(\lambda(s)) - 1] \quad (230)$$

$$\frac{s}{f(s)}\frac{\partial f(s)}{\partial s} = d_N - N\gamma_d(\lambda(s)) \,. \quad (231)$$

The last equation can be integrated to give

$$f(s) = s^{d_N}\exp\left[-\int_1^s \frac{N\gamma_d(\lambda(s'))ds'}{s'}\right] , \quad (232)$$

as can be explicitly checked:

$$\begin{aligned} s\frac{\partial f}{\partial s} &= d_N s^{d_N} \exp[\ldots] + s^{d_N} s \left(\frac{-N\gamma_d(\lambda(s))}{s} \right) \exp[\ldots] \\ &= [d_N - N\gamma_d(\lambda(s))] s^{d_N} \exp[\ldots] \\ &= [d_N - N\gamma_d(\lambda(s))] f(s) \,. \end{aligned} \tag{233}$$

Substituting the form of $f(s)$ given above into Eq. (225), we obtain

$$\Gamma^N(sp; m, \lambda, \mu) = s^{d_N = 4-N} \exp\left[-\int_1^s \frac{N\gamma_d(\lambda(s'))ds'}{s'} \right] \Gamma^N(p; m(s), \lambda(s), \mu) \tag{234}$$

where $\lambda(s)$ and $m(s)$ are solutions of the equations (229) and (230) given earlier, with implicit boundary conditions of

$$\lambda(s=1) = \lambda \,, \quad \text{and} \quad m(s=1) = m \,. \tag{235}$$

These equations are, of course, nothing more than the moving coupling constant and moving mass equations that we have already discussed where we would write $\mu = s\mu_{s=1}$ from which we see that $\mu\frac{\partial}{\partial\mu} = s\frac{\partial}{\partial s}$.

To make the boundary conditions explicit, we can write

$$\lambda(s) = \overline{\lambda}(s, \lambda), \ldots \tag{236}$$

Sometimes, it is convenient to use

$$t = \ln s, \quad \Rightarrow \quad s\frac{\partial}{\partial s} = \frac{\partial}{\partial t}, \tag{237}$$

in which case, we would have

$$\Gamma^N(sp; m, \lambda, \mu) = s^{d_{N=4-N}} \exp\left[-\int_0^t N\gamma_d(\overline{\lambda}(t', \lambda))dt'\right] \Gamma^N(p; \overline{m}(t, \lambda, m), \overline{\lambda}(t, \lambda), \mu). \tag{238}$$

In any case, our solution for $\Gamma^N(sp; m, \lambda, \mu)$ says that the N-point function at sp can be obtained from that at p by:

1. supplying naive dimensionful scaling;

2. including an extra scale dependence that comes from the wave-function renormalization associated with each external leg;

   If γ_d were a constant then $\int_0^t \gamma_d dt' = \gamma_d t = \gamma_d \ln s$ and the $\exp[\ldots] \to s^{-N\gamma_d}$.

As a result, γ_d is referred to as the anomalous dimension associated with each external leg.

3. **changing m and λ to their moving values.**

The form of $\beta(\lambda)$ and the ultimate fate of λ

Some of the possibilities for the running coupling constant can be illustrated by the sample graph for $\beta(\lambda)$ shown in Fig. 6, which might be result if one could compute exactly in the context of some theory (here we are going beyond ϕ^4 possibly).

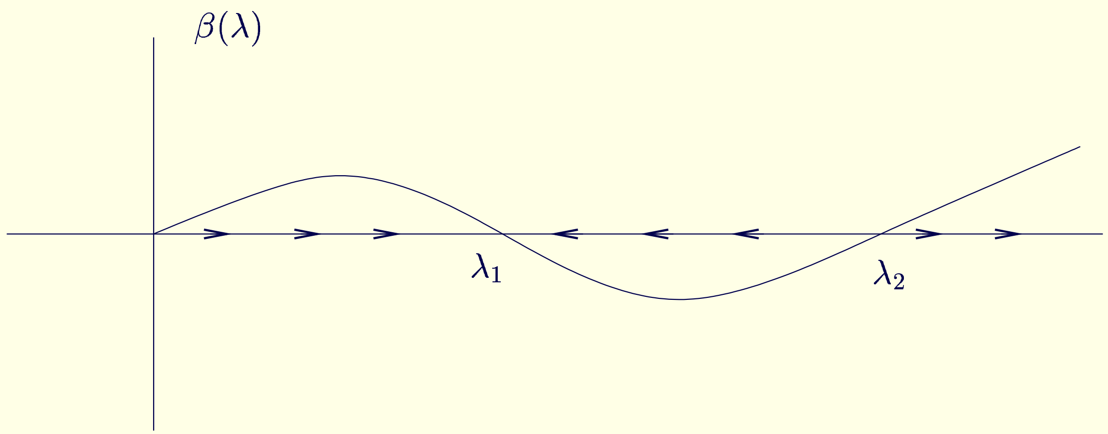


Figure 6: Graphical illustration of a possible $\beta(\lambda)$. Arrows indicate the direction of flow of λ as t *increases*.

The points $\lambda = 0, \lambda_1, \lambda_2$ where $\beta(\lambda)$ vanishes are called fixed points. If the value of λ is such that $\beta(\lambda) = 0$ then λ will not evolve any further once

λ reaches such a value. More precisely, using our notation of $\overline{\lambda}(t,\lambda)$, where $\overline{\lambda}(t=0,\lambda)=\lambda$ — i.e. the value λ referenced in the functional form is the boundary condition value, then if $\frac{d}{dt}\overline{\lambda}(t,\lambda)=\beta(\overline{\lambda})=0$ we conclude that $\overline{\lambda}(t,\lambda)$ does not change after reaching such a fixed point.

Note that since perturbation theory tells us (independent of the type of theory) that $\beta(\lambda)\to 0$ as $\lambda\to 0$, $\lambda=0$ is always a fixed point.

Furthermore, we can distinguish two types of fixed points.

1. Consider $\overline{\lambda}$ near λ_1:

   For $\overline{\lambda}<\lambda_1$, $\beta(\overline{\lambda})>0$ and so $\frac{d}{dt}\overline{\lambda}>0$ implying that $\overline{\lambda}$ is driven towards λ_1 as $t\uparrow$.

   For $\lambda_1<\overline{\lambda}<\lambda_2$, $\beta(\overline{\lambda})<0$ and so $\frac{d}{dt}\overline{\lambda}<0$, implying that $\overline{\lambda}$ is again driven towards λ_1 as $t\uparrow$.

   In short, for $0<\overline{\lambda}<\lambda_2$, $\overline{\lambda}$ is always driven $\to\lambda_1$ as t increases (i.e. as momenta become larger). Such a fixed point is called an "ultraviolet stable" fixed point.

2. Similarly, for $\overline{\lambda}$ near $\overline{\lambda}\sim 0$ or $\overline{\lambda}$ near $\overline{\lambda}\sim\lambda_2$, $\overline{\lambda}$ is driven to these values as $t\to 0$, i.e. for small momenta.

Thus, $\lambda = 0$ and $\lambda = \lambda_2$ are "infrared stable' fixed points.

Of course, if $\beta(\lambda)$ starts at 0 at $\lambda = 0$ and then becomes negative for $\lambda > 0$, then for $\overline{\lambda} > 0$ we have $\frac{d}{dt}\overline{\lambda} = \beta(\overline{\lambda}) < 0$, implying that $\overline{\lambda} \to 0$ as $t \uparrow$. In this case, $\lambda = 0$ is an ultraviolet stable fixed point.

Note that since $\beta(\lambda) = \frac{3\lambda^2}{32\pi^2} > 0$ for λ near 0, ϕ^4 is a theory for which $\lambda = 0$ is infrared stable.

It turns out that only gauge theories can yield $\beta(\lambda) < 0$ for λ near 0, and thus, only these can be "asymptotically free" (i.e. $\lambda \to 0$ for large t, i.e. large momenta).

Let us now return to our original $\beta(\lambda)$ graph. As we have seen, for $0 < \overline{\lambda} < \lambda_2$,

$$\lim_{t\to\infty} \overline{\lambda}(t,\lambda) = \lambda_1$$

$$\lim_{t\to\infty} \Gamma^N(p_i, \overline{\lambda}(t,\lambda), \overline{m}(t,m,\lambda), \mu) = \Gamma^N(p_i, \lambda_1, \overline{m}(t,m,\lambda_1), \mu) \tag{239}$$

Let us not worry about $\overline{m}$ for now, and instead focus on the effects of the presence of γ_d. Suppose $\beta(\lambda)$ has a simple 0 at λ_1 [i.e. $\beta(\lambda) = a(\lambda_1 - \lambda)$ with $a > 0$ according to the graph] and $\gamma_d(\lambda_1) \neq 0$. Then, from $\frac{d\overline{\lambda}}{dt} = a(\lambda_1 - \overline{\lambda})$

we obtain

$$\overline{\lambda}(t,\lambda) = \lambda_1 + (\lambda - \lambda_1)e^{-at}\,. \tag{240}$$

Let us use this to compute the prefactor in front of $\Gamma^N(sp:\ldots)$ in our earlier equation:

$$\Gamma^N(sp;m,\lambda,\mu) = s^{d_{N=4-N}} \exp\left[-\int_0^t N\gamma_d(\overline{\lambda}(t',\lambda))dt'\right]\Gamma^N(p;\overline{m}(t,\lambda,m),\overline{\lambda}(t,\lambda),\mu)\,. \tag{241}$$

We have [shifting to the integration variable $y = \overline{\lambda}(t',\lambda)$ and using $\frac{d}{dt'}\overline{\lambda}(t',\lambda) = \beta(\overline{\lambda}(t',\lambda))$]

$$\begin{aligned}
\int_0^t \gamma_d(\overline{\lambda}(t',\lambda))dt' &= \int_\lambda^{\overline{\lambda}} \frac{\gamma_d(y)dy}{\beta(y)} \\
&\simeq \frac{-\gamma_d(\lambda_1)}{a}\int_\lambda^{\overline{\lambda}}\frac{d\lambda'}{\lambda'-\lambda_1} \\
&= -\frac{\gamma_d(\lambda_1)}{a}\ln\left(\frac{\overline{\lambda}-\lambda_1}{\lambda-\lambda_1}\right) \\
&= \gamma_d(\lambda_1)t \\
&= \gamma_d(\lambda_1)\ln s\,.
\end{aligned} \tag{242}$$

Then, plugging in, we obtain

$$\Gamma^N(sp, \lambda, \mu) = s^{d_N - N\gamma_d(\lambda_1)} \Gamma^N(p_i, \lambda_1, \mu)\,, \tag{243}$$

implying that, in the high-momentum limit, *each* of the N fields scales with an anomalous dimension given by $\gamma_d(\lambda_1)$.

A few final remarks/reminders:

A general computation of $\beta(\lambda)$ is not possible, but near $\lambda = 0$ perturbation theory is OK, and this region, fortunately, is very useful for comparison to experiment.

For instance, for Bjorken scaling, . . . , our results indicate that to all intensive purposes the quarks behave as if almost free when considering the asymptotic limit in $p, Q^2, \ldots \to \infty$ provided $\lambda = 0$ is an ultraviolet stable fixed point. As we have said, only gauge theories such as QCD have this property.

Before proceeding onto QCD and QED, let me return to using the power series of Eqs. (181) for a more complete and extendable derivation of, for example, the form of $\beta(\lambda)$. Our equations were:

$$\lambda_0 \;=\; (\mu^2)^\epsilon \left[\lambda + \sum_{k=1}^{\infty} \frac{a_k(\lambda)}{\epsilon^k}\right]$$

$$
\begin{aligned}
m_0^2 &= m^2\left[1+\sum_{k=1}^{\infty}\frac{b_k(\lambda)}{\epsilon^k}\right] \\
Z_\phi &= \left[1+\sum_{k=1}^{\infty}\frac{c_k(\lambda)}{\epsilon^k}\right].
\end{aligned}
\tag{244}
$$

The results obtained so far to one-loop are, from Eqs. (179) and (180):

$$
a_1=\frac{3\lambda^2}{32\pi^2}, \quad b_1=\frac{\lambda}{32\pi^2}, \quad c_1=0. \tag{245}
$$

Since the coefficients inside the $[\ldots]$'s do not depend on the mass scales, m, μ, this scheme that we are employing is also called the *mass-independent* **renormalization scheme. The m, μ dependence can be avoided to arbitrarily high order.**

Intuitively, the counterterms have just the minimal structure needed to cancel ∞'s coming from big momenta in the loop integrals, and in the $momenta \to \infty$ limit, all masses are irrelevant. Mass dependence can then only appear through the finite parts of the counterterms, and we have chosen a scheme in which they do not.

To see more generally why this scheme is so useful, let us reexamine the coupling constant relation (as an example). Differentiating the top equation above with respect to μ at fixed λ_0 (just as we did before) gives

$$0 = 2\epsilon\left(\lambda + \sum_{k=1}^{\infty}\frac{a_k(\lambda)}{\epsilon^k}\right) + \mu\frac{\partial\lambda}{\partial\mu}\left(1 + \sum_{k=1}^{\infty}\frac{a_k'(\lambda)}{\epsilon^k}\right). \quad (246)$$

Now, we know that λ and $\mu\frac{\partial\lambda}{\partial\mu}$ are analytic at $\epsilon \to 0$. As a result, we can write

$$\mu\frac{\partial\lambda}{\partial\mu} = d_0(\lambda) + d_1(\lambda)\epsilon + d_2(\lambda)\epsilon^2 + \ldots. \quad (247)$$

We can then examine the preceding equation in the form:

$$2\epsilon\lambda + (2a_1 + \mu\frac{\partial\lambda}{\partial\mu}) + \sum_{k=1}^{\infty}\frac{1}{\epsilon^k}\left[\frac{da_k}{d\lambda}\mu\frac{\partial\lambda}{\partial\mu} + 2a_{k+1}\right] = 0 \quad (248)$$

keeping in mind the substitution of Eq. (247). Matching powers of ϵ gives

$$\mathcal{O}(\epsilon^{n>1}) \quad \leftrightarrow \quad d_n + \sum_{k=1}^{\infty}\frac{da_k}{d\lambda}d_{n+k} = 0 \Rightarrow d_{n\geq 2} = 0$$

$$\begin{aligned}
\mathcal{O}(\epsilon^1) &\leftrightarrow 2\lambda + d_1 = 0 \Rightarrow d_1 = -2\lambda \\
\mathcal{O}(\epsilon^0) &\leftrightarrow 2a_1 + d_0 + d_1\frac{da_1}{d\lambda} = 0 \Rightarrow d_0 = -2a_1 + 2\lambda\frac{da_1}{d\lambda} \\
\mathcal{O}(\frac{1}{\epsilon^{r\geq 1}}) &\leftrightarrow 2a_{r+1} + d_1\frac{da_{r+1}}{d\lambda} + d_0\frac{da_r}{d\lambda} = 0 \Rightarrow \ldots \qquad (249)
\end{aligned}$$

Regarding the $\epsilon^{n>1}$ result, we obtain a set of linear equations for $d_2, d_3, d_4, \ldots$, all with 0 on the right-hand side. So long as the matrix involved has an inverse, the only solution is $d_2 = d_3 = \ldots = 0$. There is no reason for the matrix defined by the $\frac{da_k}{d\lambda}$ not to have an inverse.

The most important result from the above is that from reading off the d_0 and d_1 coefficients:

$$\begin{aligned}
\mu\frac{\partial\lambda}{\partial\mu} &= d_0 + d_1\epsilon = (-2a_1 + 2\lambda\frac{da_1}{d\lambda}) - 2\lambda\epsilon \\
&\stackrel{\epsilon\to 0}{\rightarrow} -2(1 - \lambda\frac{d}{d\lambda})a_1(\lambda) \\
&= \beta(\lambda)\,. \qquad (250)
\end{aligned}$$

Note that $\beta(\lambda)$ depends only λ and is determined by the residue of the

simplest $\frac{1}{\epsilon}$ single pole term. In leading order, we have computed $a_1(\lambda) = \frac{3\lambda^2}{32\pi^2}$ from which the form of β that we have been using is obtained:

$$\begin{aligned} \beta(\lambda) &= -2(1 - \lambda\frac{d}{d\lambda})\frac{3\lambda^2}{32\pi^2} + \ldots \\ &= -2(\lambda^2 - 2\lambda^2)\frac{3}{32\pi^2} \\ &= \frac{3\lambda^2}{16\pi^2}, \end{aligned} \tag{251}$$

as obtained earlier.

The $r \geq 1$ case can be rewritten as

$$\begin{aligned} 2a_{r+1} + d_1\frac{da_{r+1}}{d\lambda} + d_0\frac{da_r}{d\lambda} &= 0 \\ \Rightarrow 2(1 - \lambda\frac{d}{d\lambda})a_{r+1} &= -d_0\frac{da_r}{d\lambda} \\ &= (2a_1 + d_1\frac{da_1}{d\lambda})\frac{da_r}{d\lambda} \end{aligned}$$

$$\begin{aligned} &= (2a_1 - 2\lambda\frac{da_1}{d\lambda})\frac{da_r}{d\lambda} \\ &= 2(1 - \lambda\frac{d}{d\lambda})a_1\frac{da_r}{d\lambda}. \end{aligned} \tag{252}$$

This result is remarkable in that it says that all the higher pole terms can be computed once the $r = 1$ single pole residue is known. This is equivalent to saying that the leading logarithms, the next-to-leading logarithms, etc. can be calculated to all orders by using the renormalization group equation with the computation of just a few low-order terms.

We can repeat this same sort of analysis for m^2. We have

$$m_0^2 = m^2 \left[b_0 + \sum_{k=1}^{\infty} \frac{b_k(\lambda)}{\epsilon^k} \right] \tag{253}$$

where $b_0 = 1$ and $b_1 = \frac{\lambda}{32\pi^2}$ (to the order we have computed). In fact, $b_0 = 1$ to all orders since it is δm^2 that has the ϵ singularities. Taking the $\mu\frac{\partial}{\partial\mu}$

derivative gives

$$0 = \mu \frac{\partial m^2}{\partial \mu} \left[b_0 + \sum_{k=1}^{\infty} \frac{b_k(\lambda)}{\epsilon^k} \right] + m^2 \left[\mu \frac{\partial b_0}{\partial \mu} + \sum_{k=1}^{\infty} \frac{\mu \frac{\partial b_k}{\partial \mu}}{\epsilon^k} \right] , \tag{254}$$

where $\mu \frac{\partial b_0}{\partial \mu} = 0$. Now since b_k depends on μ only through λ, we can rewrite this as

$$\mu \frac{\partial m^2}{\partial \mu} \left[b_0 + \sum_{k=1}^{\infty} \frac{b_k(\lambda)}{\epsilon^k} \right] = -m^2 \sum_{k=1}^{\infty} \frac{\mu \frac{\partial \lambda}{\partial \mu} \frac{d b_k}{d \lambda}}{\epsilon^k} . \tag{255}$$

At this point, we recall that

$$\mu \frac{\partial \lambda}{\partial \mu} = d_0 + d_1 \epsilon = -2(1 - \lambda \frac{d}{d\lambda}) a_1 - 2\lambda\epsilon , \tag{256}$$

and proceed with power series matching to obtain:

$$\mathcal{O}(\epsilon^0) \quad \leftrightarrow \quad \mu \frac{\partial m^2}{\partial \mu} = 2\lambda m^2 \frac{d b_1}{d\lambda} \tag{257}$$

$$\mathcal{O}(\frac{1}{\epsilon^{r \geq 1}}) \quad \leftrightarrow \quad \mu \frac{\partial m^2}{\partial \mu} b_r = -m^2 \left[-2(1 - \lambda \frac{d}{d\lambda}) a_1 \frac{d b_r}{d\lambda} - 2\lambda \frac{d b_{r+1}}{d\lambda} \right] . \tag{258}$$

where this latter equation can be rearranged by canceling a common factor of $2m^2$ to read

$$\lambda \frac{db_{r+1}}{d\lambda} = \lambda \frac{db_1}{d\lambda} b_r - \frac{db_r}{d\lambda}(1 - \lambda \frac{d}{d\lambda}) a_1 \,. \tag{259}$$

Obviously, Eq. (257) defines a running mass equation of the form:

$$\begin{aligned} \gamma_m \equiv \frac{1}{2}\mu \frac{\partial \ln m^2}{\partial \mu} &= \frac{1}{2} \frac{\mu \frac{\partial m^2}{\partial \mu}}{m^2} \\ &= \lambda \frac{db_1}{d\lambda} \\ &= \frac{\lambda}{32\pi^2} \end{aligned} \tag{260}$$

to the one-loop order computed, in agreement with our earlier result. This is supplemented with the recursion relation for the higher $\frac{1}{\epsilon}$ power singularity residues given in Eq. (259).

Finally, we have the Z_ϕ expansion:

$$Z_\phi = 1 + \sum_{k=1}^{\infty} \frac{c_k(\lambda)}{\epsilon^k} \,, \tag{261}$$

from which we compute

$$\frac{1}{2}\mu\frac{\partial Z_\phi}{\partial\mu} = \frac{1}{2}\mu\frac{\partial\lambda}{\partial\mu}\left[\frac{dc_1}{d\lambda}\frac{1}{\epsilon}+\frac{dc_2}{d\lambda}\frac{1}{\epsilon^2}+\ldots\right]$$
$$= \gamma_d\left[1+\frac{c_1}{\epsilon}+\frac{c_2}{\epsilon^2}\right] \quad (262)$$

using $\gamma_d \equiv \frac{1}{2}\mu\frac{\partial Z_\phi}{\partial\mu}Z_\phi^{-1}$. Once again, we insert

$$\mu\frac{\partial\lambda}{\partial\mu} = d_0 + d_1\epsilon \quad (263)$$

with $d_0 = \beta(\lambda)$ and $d_1 = -2\lambda$, and match powers of ϵ. The main result is the ϵ^0 coefficient matching equation:

$$\frac{1}{2}d_1\frac{dc_1}{d\lambda} = \gamma_d \quad (264)$$

which converts to

$$\gamma_d = -\lambda\frac{dc_1}{d\lambda}\,, \quad (265)$$

as quoted earlier. The more singular $\frac{1}{\epsilon^{r\geq 1}}$ matching equations give recursion relations for $c_{k\geq 2}$. For example, the $\mathcal{O}(\epsilon^{-1})$ matching gives

$$d_1\frac{dc_2}{d\lambda}+\frac{1}{2}d_0\frac{dc_1}{d\lambda}=\gamma_d c_1 \tag{266}$$

which allows a computation of c_2 once the residue of the single pole, c_1, is known.

Renormalization in Gauge Theories

The prototype here is, of course, QED. Our Feynman rules are:

$$\begin{aligned} \gamma e^- e^+ \; vertex &\rightarrow -ie\gamma_\mu \\ photon \; propagator &\rightarrow \frac{-ig_{\mu\nu}}{k^2} \\ fermion \; propagator &\rightarrow \frac{i}{\not k - m} . \end{aligned} \tag{267}$$

Degree of Divergence

Denoting by L the number of loops, IB the number of internal vector boson propagators and IF the number of internal fermion lines,

$$D = 4L - 2IB - IF \, . \tag{268}$$

We want to eliminate IB and IF in terms of external particle counts: EB, number of external photons; and EF, the number of external fermion or antifermion lines. Note that EF must be even since every incoming fermion line must exit somewhere. Let us use the notation V for the number of internal vertices, each of which comes with a $\delta^4(\ldots)$. And, there is one overall $\delta^4(\sum p_{external})$ that must pop out.

Thus, the number of independent loops is given by

$$L = IB + IF - V + 1\,. \tag{269}$$

The total number of fermion lines available to attach to vertices is $2IF + EF$ (each internal line can run from one vertex to another, but each external line has only one end to attach to a vertex). Since each vertex eats up 2 fermion lines, we have

$$2IF + EF = 2V\,, \quad or \quad IF = V - \frac{1}{2}EF\,. \tag{270}$$

Similarly, the number of photon lines available to attach to vertices is $2IB + EB$, but, each vertex accepts only one photon line. Hence,

$$2IB + EB = V\,, \quad or \quad IB = \frac{1}{2}(V - EB)\,. \tag{271}$$

Substituting L into the D expression, and then the results for IF and IB, we obtain

$$D = 2IB + 3IF - 4V + 4 = (V - EB) + (3V - \frac{3}{2}EF) - 4V + 4 = 4 - \frac{3}{2}EF - EB\,. \tag{272}$$

Generalization to QCD

If we go to QCD, we have to add the 3 and 4 gluon vertices. There are also ghost-ghost gluon vertices, but these act completely parallel to the 3-gluon vertex and need not be treated separately — for every diagram containing two internal gluons connected to two 3-gluon vertices, there is a corresponding diagram in which the two internal gluons are replaced by ghosts. (Of course, there are no external ghosts.) The important new thing to keep track of is that the 3G (and also ghost-ghost gluon) vertex is proportional to a single power of momenta.

Thus, our D expression is modified to

$$D = 4L - IF - 2IB + V_{3G} \tag{273}$$

while L is as before

$$L = IB + IF - V + 1\,, \quad V = V_{GFF} + V_{3G} + V_{4G}\,. \tag{274}$$

Substituting into D we obtain

$$D = 2IB + 3IF - 4V + 4 + V_{3G}\,. \tag{275}$$

Now, we also have, just like QED,

$$2IF + EF = 2V_{GFF}\,, \quad \rightarrow \quad IF = V_{GFF} - \frac{1}{2}EF\,, \tag{276}$$

and the generalized result

$$2IB + EB = 3V_{3G} + V_{GFF} + 4V_{4G}\,, \quad \rightarrow \quad 2IB = 3V_{3G} + V_{GFF} + 4V_{4G} - EB\,, \tag{277}$$

where the coefficients are simply the number of gluons entering a given kind of vertex.

We now substitute these results into the last expression for D to obtain:

$$D = (3V_{3G} + V_{GFF} + 4V_{4G} - EB) + (3V_{GFF} - \frac{3}{2}EF) - 4V + 4 + V_{3G}$$

$$
\begin{aligned}
&= 4(V_{3G} + V_{GFF} + V_{4G}) - 4V + 4 - EB - \frac{3}{2}EF \\
&= 4 - EB - \frac{3}{2}EF\,,
\end{aligned} \tag{278}
$$

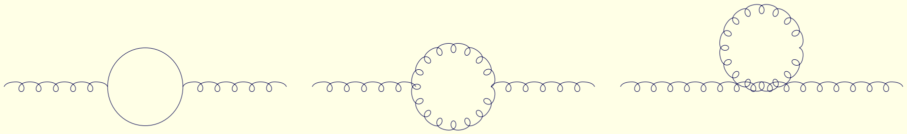
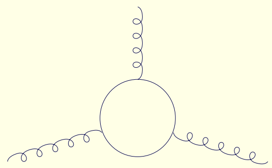
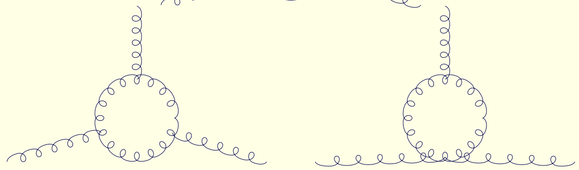
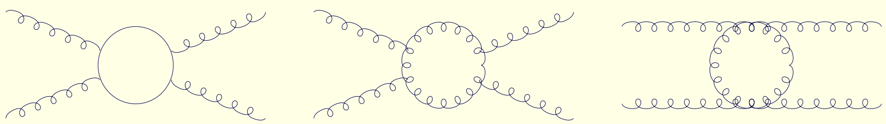
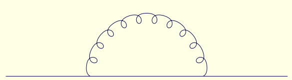
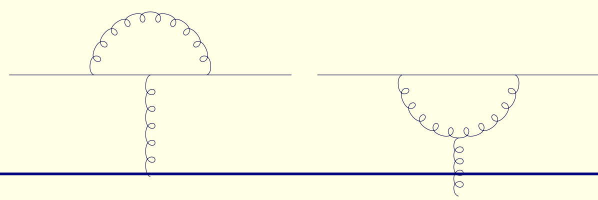
i.e. exactly the same result as in QED!

As expected, in both cases we are dealing with what we hope are renormalizable theories since they have dimensionless coupling constants (in 4 dimensions) and the degree of divergence is purely determined by the external line counts. This means that there are only a certain subset of diagrams that can be primitively divergent. Hopefully, this subset of diagrams can be counter-termed by $\Delta\mathcal{L}$ structures that are like those already present in $\mathcal{L}$.

The basic divergent diagrams in QED and QCD are summarized in the table below. The superficial degrees of divergence are those given by Eq. (278).

The actual degrees of divergence will be discussed below.

Tabulation of Divergent Diagrams (Tadpole & ghost diagrams not shown)

EF	EB	1-loop prototypes	Superficial D	Actual D
0	2		2	0 by GI
0	3		1	0 (= 0 in QED)
0	3		1	0 by GI
0	4		0	<0 by GI
2	0		1	0 by l sym.
2	1		0	0

For QED, just eliminate all diagrams with $3g$ or $4g$ vertices.

The above table does not show the diagrams in which a single photon or gluon attaches to a closed fermion loop, nor the QCD diagram in which a single gluon attaches to a closed gluon loop. The superficial D for these diagrams is $D = 3$. But, in fact the diagrams are zero. This is because they correspond to the vacuum expectation value of an electromagnetic (color) current coupling to the polarization vector for a photon (gluon). But, the vacuum expectation value of any such current must be zero if the vacuum is not to have a preferred 4-vector direction, contrary to invariance of the vacuum under Lorentz transformations. In the case of QED, the vanishing of this type of diagram can also be derived assuming that the vacuum is charge conjugation invariant. A non-zero value of the vacuum expectation value of the electromagnetic current would change sign under C. Another way of realizing that these diagrams must be zero is to note that it would be inconsistent for the $vacuum$ to carry a net electromagnetic or color current. In fact, as discussed momentarily, it is easy at one-loop to explicitly show that these diagrams are zero.

In general, there are many subtleties that can cause the actual degree of divergence to differ from the superficial value of D. First is the fact that sometimes the naive degree of divergence is reduced by virtue of gauge invariance requirements on the structure of the diagram or by symmetry. The important cases are:

1. **the bubble insertions into a gluon or photon propagator.**

   Gauge invariance demands that the sum of the diagrams be proportional to $k_\mu k_\nu - k^2 g_{\mu\nu}$, i.e. there are two powers of $external$ momentum that must pop out, thereby decreasing the value of D by 2 units. Actually, this works separately for fermion and gluon loops.

2. **the gauge loop correction to a fermion propagator.**

   We can understand easily at one loop what the issue is. Consider very large values of the loop integration momentum l. Since the internal fermion propagator goes like $1/\not{l}$ while the internal gauge propagator goes like $1/l^2$, the divergence would be $\int d^4 l \frac{\not{l}}{(l^2)^2}$ which is odd under $l \to -l$. Thus, the leading term cannot survive. Instead of being linearly divergent, the diagram is only log divergent.

3. **the loop corrections to the four-gluon vertex.**

   Again, there is a gauge requirement that means this correction must be proportional to appropriate gauge structures for the external gluons. Roughly, one must have a momentum structure that is the Fourier transform of $(F_{\mu\nu})^4$, a dimension 8 form. So, even though $D = 0$ superficially, in fact $D < 0$. This works separately for fermion and gluon loops.

4. the fermion loop possibility for a three-photon or a three-gluon vertex.

In the three-photon case, there is the theorem that the fermion-loop correction must be 0 by virtue of charge conjugation symmetry (Furry's theorem). This is also needed for renormalizability in the sense that if we had a divergence for this graph, then the counter term (a 3 photon vertex) would correspond to an interaction that is not present in the QED Lagrangian. This goes "counter" to the counter-term Lagrangian approach. In fact, it would mean a break down in gauge invariance; there should be no self-interactions of photons regardless of the order of perturbation theory considered.

In QCD, the fermion loop correction to the three-gluon vertex is non-zero. The actual D is 0 by virtue of the leading contribution being odd under $l \to -l$ for the loop integration, $\int d^4l \left(\frac{1}{\not{l}}\right)^3$.

5. the gluon loop corrections to the three-gluon vertex.

Here, gauge invariance structure requirements for the vertex (which means it must look like our bare vertex which was linear in *external* momenta arranged in the appropriate antisymmetrized form — constructed so that $p^\alpha \epsilon^\beta \epsilon^\gamma V_{\alpha\beta\gamma}(p,q,r) = \ldots = 0$ for example), reduce the degree of divergence from 1 to 0.

The second subtlety is that the divergence of a diagram can actually be larger than D when it contains divergent subdiagrams corresponding to the basic primitive divergences isolated above and discussed below. We will return to this point later.

For now, let us consider just our basic primitively divergent types of graphs. Before considering ghosts, we have:

1. gluon (or photon) propagator corrections;

2. fermion propagator corrections;

3. gluon (or photon) fermion-antifermion vertex corrections;

4. three-gluon vertex corrections (but not three-photon vertex corrections, since these are 0 by virtue of Furry's theorem).

Well, I hope it is obvious that all these divergences can be compensated by counter terms of the same form as terms already present in the bare Lagrangian.

After including ghosts, the additional divergent items are:

1. ghost propagator corrections;

2. gluon-ghost-ghost vertex corrections.

Again, the necessary counterterms have the same form as terms already present in the ghost addition to the bare Lagrangian.

Another interesting point will emerge once we consider dimensional regularization. One finds that the propagator tadpole diagrams are 0 for massless virtual propagators when regularized via dimensional regularization. This is another great advantage of dimensional regularization. This feature is related to the fact that dimensional regularization preserves gauge invariance. This is something that we shall return to once we do explicit calculations.

Let us next briefly return to discuss why, at one loop, the diagrams with one external photon or gluon are simply zero. In the case of QED, we have

$$\epsilon^\mu \int d^4 l \frac{\mathrm{Tr}[\gamma_\mu (\not{l} + m)]}{l^2 - m^2} = \epsilon^\mu \int d^4 l \frac{4 l_\mu}{l^2 - m^2} = 0 \tag{279}$$

by $l \to -l$ symmetry. The same applies to the QCD fermion loop diagram. The QCD gluon loop diagram is zero partly because of similar symmetry arguments, but also requires using the same result for a $massless$ gluon propagator in the internal loop as stated in the previous paragraph.

Next, let us return to the fact that a diagram can have $D < 0$, for example, and actually be divergent. An example of this kind is photon exchange between two scattering electrons in which a fermion loop is inserted in the exchanged photon. Naively, $D = -2$ for this diagram ($EF = 4$, $EB = 0$), but the fermion loop insertion actually gives a $D = 0$ logarithmic divergence. This, however, is rendered finite once the photon propagator counter term component of $\Delta\mathcal{L}$ is introduced. So, the value of a diagram's superficial divergence D only applies once all divergent subdiagrams (at lower order in perturbation theory) have been rendered finite by the counterterm $\Delta\mathcal{L}$.

Before turning to the explicit calculations, it's useful to discuss a couple of additional things. First, let us understand a little bit more about non-renormalizable cases. Suppose we lived in 6 dimensions instead of 4. Then, sticking to QED for simplicity (but a similar result appears in QCD), we would have

$$\begin{aligned} D &= 6L - IF - 2IB = 4IB + 5IF - 6V + 6 \\ &= (2V - 2EB) + (5V - \frac{5}{2}EF) - 6V + 6 \\ &= V + 6 - 2EB - \frac{5}{2}EF \,. \end{aligned} \tag{280}$$

We see that D increases with the vertex number V which we know is a problem. To be more specific, suppose we had a diagram with 2 incoming fermions, 2 outgoing fermions, one incoming photon and one outgoing photon. Suppose we constructed a $V = 8$ diagram for this case, being careful not to choose a diagram with subdiagrams having a higher degree of superficial divergence. One such diagram is constructed as follows. Take the two external photons and attach them to a closed fermion loop which then emits two virtual photons (#1 and #2). The first of these emits an external fermion (#1) and a virtual fermion, the latter turning into a 2nd external fermion (#2) and another (#3) virtual photon. Virtual photons #2 and #3 then both attach to a fermion line that begins with external fermion #3 and ends with external fermion #4, with a virtual fermion propagator between the attachments of the #2 and #3 virtual photons. This diagram would have $D = 0$. A counter term to correct for this would have the structure $(\overline{\psi}\psi)^2 A^2$ and/or $(\overline{\psi}\not{A}\psi)^2$ (I have not worked it out) with infinite coefficient constructed so that the $\frac{1}{\epsilon}$ structure of the counterterm cancels the $\frac{1}{\epsilon}$ of the explicit diagram. Well, obviously such a counterterm is not a structure that appears in the bare Lagrangian.

The moral is that one would have to introduce counter terms for diagrams with any arbitrarily large number of external fermions and photons. This is characteristic of a non-renormalizable theory.

Finally, we want to consider the connection between renormalizability and dimensionless coupling constant: gauge theory version.

Consider a QED-like theory with just one type of interaction or vertex, and define:

- B = # of boson lines entering or exiting a vertex;
- F = # of fermion lines entering or exiting a vertex;
- d= # of derivatives associated with vertex interaction.

Then we can generalize our earlier results as follows.

$$\begin{aligned} 2IF + EF &= FV \\ 2IB + EB &= BV \end{aligned} \tag{281}$$

and as before the number of loops is

$$L = IB + IF - V + 1\,, \tag{282}$$

from which we derive (noting that each derivative at a vertex gives a momentum power upstairs — just like the 3-gluon vertex example)

$$\begin{aligned} D &= 2IB + 3IF - 4V + dV + 4 \\ &= (BV - EB) + \frac{3}{2}(VF - EF) + dV - 4V + 4 \\ &= 4 - \frac{3}{2}EF - EB + V(B + \frac{3}{2}F + d - 4) \\ &\equiv 4 - \frac{3}{2}EF - EB + V\delta V\,. \end{aligned} \tag{283}$$

As we have come to understand, standard renormalizability corresponds to $\delta V = 0$. We will now show that the dimensionality of the coupling constant associated with the vertex is related to δV. Let us recall:

- dim$[\psi]$ =dim$[\overline{\psi}] = M^{3/2}$;
- dim$[A] = M^1$;
- dim of a derivative $= M^1$.

Then, since

$$\int d^4x g_{vertex} (\overline{\psi} \text{ or } \psi)^F (A)^B (\partial_x)^d = M^0 \tag{284}$$

we must have

$$M^{-4} (M)^{\text{dim}[g]} M^{3F/2} M^B M^d = M^0 \tag{285}$$

or, equivalently,

$$\text{dim}[g] = 4 - B - \frac{3}{2} F - d = -\delta V \,. \tag{286}$$

From this result, we that in a vector fermion theory such as QED (also QCD) only a theory with dimensionless coupling constant(s) is renormalizable.

If $\delta V > 0$, then D increases with the number of vertices V so that higher order diagrams are increasingly divergent and a process involving arbitrary numbers of external particles will be divergent if a sufficiently high order of perturbation is considered. In this case, the coupling constant has negative mass dimension $M^{-\delta V}$. This is the case of non-renormalizability.

If $\delta V < 0$, then D decreases with the number of vertices, implying that only some lower order diagrams will be divergent. The coupling constant has positive mass dimension $M^{-\delta V} = M^{|\delta V|}$ in this case. This is the case of super-renormalizability.

Explicit 1-loop renormalization and computations for QED

At the moment I plan to follow Ryder to a large extent, so that will be a useful reference.

We will carry out the procedure in the Feynman version of the Lorentz gauge, for which the appropriate $\mathcal{L}$ is

$$\begin{aligned}\mathcal{L} &= i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi - eA^\mu\overline{\psi}\gamma_\mu\psi \\ &\quad -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2}(\partial_\mu A^\mu)(\partial_\nu A^\nu)\,. \end{aligned} \tag{287}$$

First, we must do some dimensional analysis to see what the dimensions of things are in n dimensions in order that the mass dimension $\mathrm{dim}[\int d^n x\mathcal{L}] = 0$.

- From $i\overline{\psi}\gamma^\mu\partial_\mu\psi$ we determine that $\mathrm{dim}[\psi] = \frac{n-1}{2}$ so that

$$\mathrm{dim}[i\overline{\psi}\gamma^\mu\partial_\mu\psi] = 2\frac{n-1}{2} + 1 = n \tag{288}$$

  cancels $\mathrm{dim}[d^n x] = -n$.

- If $\dim[m] = 1$ then the $m\overline{\psi}\psi$ form will also have net dimension of n as required.

- From the $-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$ part of $\mathcal{L}$ we find that $\dim[A] = \frac{n}{2} - 1$.

- This is consistent also for the gauge fixing term of $\mathcal{L}$.

- Thus,

$$\dim[-eA^\mu\overline{\psi}\gamma_\mu\psi] = \dim[e] + 2\frac{n-1}{2} + \frac{n}{2} - 1 = n \tag{289}$$

  requires

$$\dim[e] = 2 - \frac{n}{2}. \tag{290}$$

  Thus, in close analogy to what we did in ϕ^4 theory, we will replace

$$e \to e(\mu)^{2-\frac{n}{2}} = e\mu^\epsilon. \tag{291}$$

As regards external spinors, e.g. $u(p), \ldots$, these may be kept as in $n = 4$ dimensions. We only need to go to $n \neq 4$ dimensions for stuff in the loop computations.

This brings of the question of how we deal with Dirac algebra and matrices in n dimensions. There are actually two approaches.

1. **Dimensional regularization, which is the approach we follow and in which there will be some changes from 4 dimensions.**

2. **Dimensional reduction, in which the numerator algebra is done as if in 4 dimensions and only the non-Dirac part of the propagators are treated in n dimensions.**

At one-loop the two approaches give the same results. There is an ongoing debate as to the number of loops to which this equivalence extends.

In dimensional regularization, we modify the Dirac algebra as follows. We begin with the anticommutator

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \tag{292}$$

where we consider $g_{\mu\nu}$ to be defined in n dimensions so that $\delta^\mu_\mu = n$. For consistency, this then implies that

$$\gamma^\mu\gamma_\mu = n, \quad \gamma_\mu\gamma^\nu\gamma^\mu = (2-n)\gamma^\nu\,, \quad \gamma_\rho\gamma_\mu\gamma_\nu\gamma^\rho = 4g_{\mu\nu} + (n-4)\gamma_\mu\gamma_\nu\,. \tag{293}$$

We will also define

$$\mathrm{Tr}[I] = 2^{n/2} \tag{294}$$

(any regular function approaching 4 as $n \to 4$ would have been ok) for which consistency with $\gamma^\mu \gamma_\mu = n$ then requires

$$\mathrm{Tr}[\gamma_\mu \gamma_\nu] = 2^{n/2} g_{\mu\nu}\,, \quad \mathrm{Tr}[\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma] = 2^{n/2}[g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}]\,. \tag{295}$$

However, γ_5 becomes a problem in this approach as we need all 4 gamma matrices to define it. In our QED renormalization we don't have to face this problem, but in parity-violating theories we do. This is a long story and is related to "anomalies" that can destroy the renormalizability of a theory. We don't have time to go into this during these lectures.

So now let us do a few QED calculations.

- **The photon loop correction to the fermion propagator.**

$$\begin{aligned} -i\Sigma(p) &= (-ie\mu^\epsilon)^2 \int \frac{d^n k}{(2\pi)^n} \gamma^\mu \frac{i}{\not p - \not k - m} \frac{-i g_{\mu\nu}}{k^2} \gamma^\nu \\ &= -e^2 \mu^{2\epsilon} \int \frac{d^n k}{(2\pi)^n} \frac{\gamma_\mu (\not p - \not k + m)\gamma^\mu}{[(p-k)^2 - m^2]k^2} \\ &= -e^2 \mu^{2\epsilon} \int d\alpha \int \frac{d^n k}{(2\pi)^n} \frac{\gamma_\mu (\not p - \not k + m)\gamma^\mu}{[\alpha(p-k)^2 - \alpha m^2 + (1-\alpha)k^2]^2} \end{aligned}$$

$$
\begin{aligned}
&= -e^2\mu^{2\epsilon}\int d\alpha \int \frac{d^nk'}{(2\pi)^n}\frac{\gamma_\mu[(1-\alpha)\not p - \not k' + m]\gamma^\mu}{[k'^2 - \alpha m^2 + \alpha(1-\alpha)p^2]^2} \\
&= -e^2\mu^{2\epsilon}\int d\alpha \gamma_\mu[(1-\alpha)\not p + m]\gamma^\mu \int \frac{d^nk'}{(2\pi)^n}\frac{}{[k'^2 - \alpha m^2 + \alpha(1-\alpha)p^2]^2} \qquad (296)
\end{aligned}
$$

In the above, we introduced $k' = k - \alpha p$ in order to write the denominator in a diagonal form, i.e. no terms of the form $k' \cdot p$. At this point, the $\not k'$ disappears because it is odd under $k' \to -k'$, and the integral then reduces to a logarithmically divergent integral. We now do the same thing we did before of performing a Wick rotation on the k'_0 integration and then reducing the integration to something that corresponds to a certain Γ function. The result is (where we use $n = 4 - 2\epsilon$ at a certain point — don't forget that Ryder uses $n = 4 - \epsilon$ if you are looking at his stuff):

$$
\begin{aligned}
-i\Sigma(p) &= -ie^2\mu^{2\epsilon}\frac{\Gamma(2-n/2)}{(4\pi)^{n/2}}\int d\alpha \gamma_\mu[(1-\alpha)\not p + m]\gamma^\mu[\alpha m^2 - \alpha(1-\alpha)p^2]^{n/2-2} \\
&= -ie^2\mu^{2\epsilon}\frac{\Gamma(2-n/2)}{(4\pi)^{n/2}}\int d\alpha [(1-\alpha)\not p(2-n) + mn][\alpha m^2 - \alpha(1-\alpha)p^2]^{n/2-2} \\
&= -ie^2\frac{\Gamma(\epsilon)}{16\pi^2}\int d\alpha[(1-\alpha)\not p(-2+2\epsilon) + m(4-2\epsilon)]\left[\frac{\alpha m^2 - \alpha(1-\alpha)p^2}{4\pi\mu^2}\right]^{-\epsilon} \\
&= -ie^2\frac{(\frac{1}{\epsilon}-\gamma)}{16\pi^2}\int d\alpha[(1-\alpha)\not p(-2+2\epsilon) + m(4-2\epsilon)]\left[\frac{\alpha m^2 - \alpha(1-\alpha)p^2}{4\pi\mu^2}\right]^{-\epsilon} \\
&= -ie^2\frac{(\frac{1}{\epsilon}-\gamma)}{16\pi^2}\int d\alpha[(1-\alpha)\not p(-2+2\epsilon) + m(4-2\epsilon)]\left[1 - \epsilon\ln\left(\frac{\alpha m^2 - \alpha(1-\alpha)p^2}{4\pi\mu^2}\right)\right]
\end{aligned}
$$

$$= \quad -i\frac{e^2}{16\pi^2\epsilon}(-\not p + 4m) - i\frac{e^2}{16\pi^2}\Bigg\{ \not p(1+\gamma) - 2m(1+2\gamma)$$

$$+2\int d\alpha[\not p(1-\alpha) - 2m]\ln\left(\frac{\alpha m^2 - \alpha(1-\alpha)p^2}{4\pi\mu^2}\right)\Bigg\} \tag{297}$$

where the singularity of interest is contained entirely in the first term.

Derivation of the integral in a general form

The above result can be read off of our old ϕ^4 case, but it is useful to derive a more general result for later use. We consider an integral over l and then Wick rotate the l_0 component to $i\bar{l}_0$, so that $l^2 \to -(\bar{l}_0^2 + l_1^2 + \ldots) = -\vec{l}_E^2$. Using these manipulations below we have the following sequence of steps:

$$\begin{aligned}
\int \frac{d^n l}{(2\pi)^n}\frac{1}{[l^2 - a^2]^A} &= i\int \frac{d^n l_E}{(2\pi)^n}\frac{1}{[-\vec{l}_E^2 - a^2]^A} \\
&= i\frac{2\pi^{n/2}}{(2\pi)^n\Gamma\left(\frac{n}{2}\right)}\int_0^\infty \frac{l^{n-1}dl}{[-l^2 - a^2]^A} \quad \text{with } l = |\vec{l}_E|
\end{aligned}$$

$$
\begin{aligned}
&= i\frac{2\pi^{n/2}}{(2\pi)^n\Gamma\left(\frac{n}{2}\right)}(-1)^A\int_0^\infty \frac{l^{n-1}dl}{[l^2+a^2]^A} \\
&= i\frac{2\pi^{n/2}}{(2\pi)^n\Gamma\left(\frac{n}{2}\right)}(-1)^A\int_0^\infty \frac{dt}{2\sqrt{t}}\frac{t^{\frac{n-1}{2}}}{[t+a^2]^A} \\
&= i\frac{\pi^{n/2}}{(2\pi)^n\Gamma\left(\frac{n}{2}\right)}(-1)^A\int_0^\infty dt\frac{t^{\frac{n}{2}-1}}{[t+a^2]^A} \\
&= i\frac{\pi^{n/2}}{(2\pi)^n\Gamma\left(\frac{n}{2}\right)}(-1)^A\frac{1}{[a^2]^{A-\frac{n}{2}}}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(A-\frac{n}{2}\right)}{\Gamma(A)} \\
&= i\frac{\pi^{n/2}}{(2\pi)^n}(-1)^A\frac{1}{[a^2]^{A-\frac{n}{2}}}\frac{\Gamma\left(A-\frac{n}{2}\right)}{\Gamma(A)} \\
&= i(-1)^{n/2}\frac{1}{(4\pi)^{n/2}}\frac{1}{[-a^2]^{A-\frac{n}{2}}}\frac{\Gamma\left(A-\frac{n}{2}\right)}{\Gamma(A)} \\
&= i(-1)^A\frac{1}{(4\pi)^{n/2}}\frac{1}{[a^2]^{A-\frac{n}{2}}}\frac{\Gamma\left(A-\frac{n}{2}\right)}{\Gamma(A)} \qquad (298)
\end{aligned}
$$

- **The photon propagator correction**

Next, we have the correction to the photon propagator coming from the fermion loop diagram. For reasons that will become apparent after we do the usual geometric sum of iterated 1PI diagrams, we wish to denote it by $i\Pi_{\mu\nu}$, where the μ and ν indices refer to the Lorentz indices of the photons entering and exiting the diagram. The form is (remembering the minus sign for a closed fermion loop and shifting to $p' = p - \alpha k$ at a certain stage):

$$
\begin{aligned}
i\Pi_{\mu\nu} &= -(-ie\mu^{\epsilon})^2 \int \frac{d^n p}{(2\pi)^n} \mathrm{Tr}\left(\gamma_\mu \frac{i}{\not p - m} \gamma_\nu \frac{i}{\not p - \not k - m}\right) \\
&= -e^2 \mu^{2\epsilon} \int \frac{d^n p}{(2\pi)^n} \frac{\mathrm{Tr}\left[\gamma_\mu(\not p + m)\gamma_\nu(\not p - \not k + m)\right]}{(p^2 - m^2)[(p-k)^2 - m^2]} \\
&= -e^2 \mu^{2\epsilon} \int d\alpha \int \frac{d^n p'}{(2\pi)^n} \frac{\mathrm{Tr}\left[\gamma_\mu(\not p' + \alpha \not k + m)\gamma_\nu(\not p' - (1-\alpha)\not k + m)\right]}{[p'^2 - m^2 + k^2\alpha(1-\alpha)]^2}
\end{aligned}
\tag{299}
$$

At this point, let us work on simplifying the numerator. We must remember that terms that are odd in p' will integrate to 0. Further, we must recall that the trace of an odd number of γ matrices is 0. Using these two ingredients, we obtain:

$$
\begin{aligned}
N &= [p'^{\rho} p'^{\sigma} - k^\rho k^\sigma \alpha(1-\alpha)]\mathrm{Tr}[\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma] + m^2 \mathrm{Tr}[\gamma_\mu\gamma_\nu] \\
&= [p'^{\rho} p'^{\sigma} - k^\rho k^\sigma \alpha(1-\alpha)] 2^{n/2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma} + g_{\mu\sigma}g_{\nu\rho}) + m^2 2^{n/2} g_{\mu\nu}] \\
&= 2^{n/2}\left\{2p'_\mu p'_\nu - 2\alpha(1-\alpha)(k_\mu k_\nu - k^2 g_{\mu\nu}) - g_{\mu\nu}[p'^2 + k^2\alpha(1-\alpha) - m^2]\right\}
\end{aligned}
\tag{300}
$$

where we organized the $k^2 g_{\mu\nu}$ term in to two pieces so that the coefficient of the final $g_{\mu\nu}$ piece above has precisely the same form as the denominator.

An aside on some useful integrals

Now we need some theorems for integrals containing integrated momenta in the numerator. What we, in particular, wish to show is that the $2p'_\mu p'_\nu$ terms cancels the $-g_{\mu\nu}[p'^2 + k^2\alpha(1-\alpha) - m^2]$ term.

A good approach is to begin with what we have already shown, which I write using a general notation.

$$\int \frac{d^n l}{(2\pi)^n} \frac{1}{[l^2 + M^2 + 2l\cdot q]^A} = \int \frac{d^n l}{(2\pi)^n} \frac{1}{[(l+q)^2 + M^2 - q^2]^A} = i(-1)^{n/2} \frac{\Gamma(A - \frac{n}{2})}{(4\pi)^{n/2}\Gamma(A)} \frac{1}{(M^2 - q^2)^{A - \frac{n}{2}}}, \quad (301)$$

where the above expression follows from Eq. (298) using the identification of $-a^2 = M^2 - q^2$.

Starting from this expression, we may generate other useful expressions by differentiation

$$\int \frac{d^n l}{(2\pi)^n} \frac{l_\mu}{[l^2 + M^2 + 2l\cdot q]^A} = -\frac{1}{2}\frac{1}{A-1}\frac{\partial}{\partial q^\mu} \int \frac{d^n l}{(2\pi)^n} \frac{1}{[l^2 + M^2 + 2l\cdot q]^{A-1}}$$

$$
\begin{aligned}
&= -\frac{1}{2}\frac{1}{A-1}\frac{\partial}{\partial q^\mu}\left[i(-1)^{n/2}\frac{\Gamma(A-1-\frac{n}{2})}{(4\pi)^{n/2}\Gamma(A-1)}\frac{1}{(M^2-q^2)^{A-1-\frac{n}{2}}}\right] \\
&= -i(-1)^{n/2}\frac{1}{2}\frac{\Gamma(A-1-\frac{n}{2})}{(4\pi)^{n/2}\Gamma(A)}[-(A-1-\frac{n}{2})](-2q_\mu)\frac{1}{(M^2-q^2)^{A-\frac{n}{2}}} \\
&= -i(-1)^{n/2}\frac{\Gamma(A-\frac{n}{2})}{(4\pi)^{n/2}\Gamma(A)}\frac{q_\mu}{(M^2-q^2)^{A-\frac{n}{2}}}\,. \qquad (302)
\end{aligned}
$$

From this result, we may proceed to

$$
\begin{aligned}
&\int \frac{d^n l}{(2\pi)^n}\frac{l_\mu l_\nu}{[l^2+M^2+2l\cdot q]^A} \\
&= -\frac{1}{2}\frac{1}{A-1}\frac{\partial}{\partial q^\mu}\int\frac{d^n l}{(2\pi)^n}\frac{l_\nu}{[l^2+M^2+2l\cdot q]^{A-1}} \\
&= -\frac{1}{2}\frac{1}{A-1}\frac{\partial}{\partial q^\mu}\left[i(-1)^{n/2}\frac{-\Gamma(A-1-\frac{n}{2})}{(4\pi)^{n/2}\Gamma(A-1)}\frac{q_\nu}{(M^2-q^2)^{A-1-\frac{n}{2}}}\right] \\
&= i(-1)^{n/2}\frac{1}{2}\frac{\Gamma(A-1-\frac{n}{2})}{(4\pi)^{n/2}\Gamma(A)}\left\{[-(A-1-\frac{n}{2})](-2q_\mu)q_\nu\frac{1}{(M^2-q^2)^{A-\frac{n}{2}}}+\frac{g_{\mu\nu}}{(M^2-q^2)^{A-1-\frac{n}{2}}}\right\} \\
&= i(-1)^{n/2}\frac{1}{(4\pi)^{n/2}\Gamma(A)}\left[\frac{1}{2}g_{\mu\nu}\frac{\Gamma(A-1-\frac{n}{2})}{(M^2-q^2)^{A-1-\frac{n}{2}}}+\frac{\Gamma(A-\frac{n}{2})q_\mu q_\nu}{(M^2-q^2)^{A-\frac{n}{2}}}\right]\,. \qquad (303)
\end{aligned}
$$

Back to the main $\Pi_{\mu\nu}$ calculation

The case of current interest, referring back to our expression for the numerator N of Eq. (300), is $l = p'$, $q = 0$, $M^2 = k^2\alpha(1-\alpha) - m^2$ and $A = 2$, for which we have

$$\int \frac{d^n l}{(2\pi)^n} \frac{2 l_\mu l_\nu}{[l^2 + M^2]^2} = i(-1)^{n/2} 2 \frac{1}{(4\pi)^{n/2}\Gamma(2)} \left[\frac{1}{2} g_{\mu\nu} \frac{\Gamma(1-\frac{n}{2})}{(M^2)^{1-\frac{n}{2}}} \right]$$
$$= i(-1)^{n/2} \frac{1}{(4\pi)^{n/2}} \left[g_{\mu\nu} \frac{\Gamma(1-\frac{n}{2})}{(M^2)^{1-\frac{n}{2}}} \right], \qquad (304)$$

which we wish to compare to the explicit $g_{\mu\nu}$ term of N which is reduced as follows:

$$-g_{\mu\nu} \int \frac{d^n l}{(2\pi)^n} \frac{l^2 + M^2}{[l^2 + M^2]^2} = -g_{\mu\nu} \int \frac{d^n l}{(2\pi)^n} \frac{1}{[l^2 + M^2]^1}$$
$$= -g_{\mu\nu} i(-1)^{n/2} \frac{1}{(4\pi)^{n/2}\Gamma(1)} \left[\frac{\Gamma(1-\frac{n}{2})}{(M^2)^{1-\frac{n}{2}}} \right]$$
$$= -i(-1)^{n/2} \frac{1}{(4\pi)^{n/2}} \left[\frac{g_{\mu\nu}\Gamma(1-\frac{n}{2})}{(M^2)^{1-\frac{n}{2}}} \right], \qquad (305)$$

which does indeed cancel the $2p'_\mu p'_\nu = 2l_\mu l_\nu$ term. Of course, there were simpler ways to get this one result, but the above general procedure will be useful in other computations.

So, now the remainder of the $i\Pi_{\mu\nu}$ expression is

$$
\begin{aligned}
i\Pi_{\mu\nu}(k) &= -e^2\mu^{2\epsilon}2^{n/2}[k_\mu k_\nu - k^2 g_{\mu\nu}]\int d\alpha \int \frac{d^n p'}{(2\pi)^n}\left\{-\frac{2\alpha(1-\alpha)}{[p'^{\,2}+k^2\alpha(1-\alpha)-m^2]^2}\right\} \\
&= -e^2\mu^{2\epsilon}2^{2-\epsilon}[k_\mu k_\nu - k^2 g_{\mu\nu}]\int d\alpha \left\{-2\alpha(1-\alpha)i(-1)^{2-\epsilon}\frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(2)}\frac{1}{[k^2\alpha(1-\alpha)-m^2]^\epsilon}\right\} \\
&= ie^2\mu^{2\epsilon}2^{2-\epsilon}[k_\mu k_\nu - k^2 g_{\mu\nu}]\int d\alpha \left\{2\alpha(1-\alpha)\frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(2)}\frac{1}{[m^2-k^2\alpha(1-\alpha)]^\epsilon}\right\} \\
&= i\frac{e^2}{2\pi^2}[k_\mu k_\nu - k^2 g_{\mu\nu}]\int d\alpha \left\{\alpha(1-\alpha)\Gamma(\epsilon)\left[\frac{m^2-k^2\alpha(1-\alpha)}{2\pi\mu^2}\right]^{-\epsilon}\right\} \\
&= i\frac{e^2}{2\pi^2}[k_\mu k_\nu - k^2 g_{\mu\nu}]\left\{\frac{1}{6}\left(\frac{1}{\epsilon}-\gamma\right) - \int d\alpha\alpha(1-\alpha)\ln\left[\frac{m^2-k^2\alpha(1-\alpha)}{2\pi\mu^2}\right] + \mathcal{O}(\epsilon)\right\} ,
\end{aligned}
\tag{306}
$$

where the $\ln[\ldots]$ came from $\frac{1}{\epsilon}(-\epsilon\ln[\ldots])$. You will notice some differences between this and Ryder's expression. Partly it is $2\epsilon_{me} = \epsilon_{Ryder}$, partly because I used what Ryder calls $f(d) = 2^{n/2}$, whereas he set $f(d) = 1$, and partly because I believe he has the wrong sign for the $m^2 - k^2\alpha(1-\alpha)$ after performing the Wick rotation and properly accounting for the $(-1)^{n/2} = (-1)^{2-\epsilon} = (-1)^{-\epsilon}$ which can be absorbed in the form

$$(-1)^{-\epsilon}[k^2\alpha(1-\alpha) - m^2]^{-\epsilon} = [m^2 - k^2\alpha(1-\alpha)]^{-\epsilon} . \tag{307}$$

Please let me know if you disagree with my sign after checking this through. There was a similar discrepancy in the sign of the $\ln$ argument in the $-i\Sigma$ expression for exactly the same reason. Incidentally, my signs agree with those in Ramond's QFT textbook, Eqs. (8.2.20) and (8.2.32), once you realize that he is using a Euclidean notation so that to compare to my expressions you must take $\not{p}_E \to -\not{p}$ and $p_E^2 \to -p^2$ where p_E is the Euclidean momentum employed by Ramond (called p in his book).

The most important property of the result of Eq. (299) is the $[k_\mu k_\nu - k^2 g_{\mu\nu}]$ structure. This is the form required by gauge invariance in the form $k^\mu \Pi_{\mu\nu} = 0$. In fact, this structure is the only form that $\Pi_{\mu\nu}$ could have and obey the gauge invariance requirement. It is because we had to get proportionality to this structure that the naive $D = 2$ divergence level was reduced to $D = 0$. Two of the momenta powers were eaten up in creating this form.

We will see that the above GI structure is also critical in order to retain zero mass for the photon after including one-loop corrections and performing dimensional regularization. You should also know that not every regularization procedure produces this kind of explicitly gauge invariant form.

We shall return to these issues shortly. But first, we need to compute

- **The vertex correction.**

The result of the photon exchange contribution to the vertex will be denoted by $-ie\Lambda_\mu$ in 4 dimensions or, in $n = 4 - 2\epsilon$ dimensions we will pull out the necessary mass scale and write $-ie\mu^\epsilon\Lambda_\mu$. We have from the diagram (using q for the momentum on the photon, p' for the outgoing electron momentum, and p for the incoming electron momentum):

$$\begin{aligned} -ie\mu^\epsilon\Lambda_\mu &= (-ie\mu^\epsilon)^3 \int \frac{d^nk}{(2\pi)^n} \frac{-ig^{\nu\rho}}{k^2} \gamma_\nu \frac{i}{\not{p}' - \not{k} - m} \gamma_\mu \frac{i}{\not{p} - \not{k} - m} \gamma_\rho \\ &= -(e\mu^\epsilon)^3 \int \frac{d^nk}{(2\pi)^n} \frac{\gamma_\nu(\not{p}' - \not{k} + m)\gamma_\mu(\not{p} - \not{k} + m)\gamma^\nu}{k^2[(p'-k)^2 - m^2][(p-k)^2 - m^2]} \\ &= -2(e\mu^\epsilon)^3 \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int \frac{d^nk}{(2\pi)^n} \frac{\gamma_\nu(\not{p}' - \not{k} + m)\gamma_\mu(\not{p} - \not{k} + m)\gamma^\nu}{[k^2 - m^2(\alpha+\beta) - 2k\cdot(\alpha p + \beta p') + \alpha p^2 + \beta p'^2]^3}, \end{aligned} \tag{308}$$

where we used the Feynman parameter trick for three denominators in the form:

$$\frac{1}{abc} = 2\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{1}{[a(1-\alpha-\beta)+\alpha b+\beta c]^3}\,. \tag{309}$$

We will now shift to $k' = k - \alpha p - \beta p'$ in order to diagonalize the denominator above so that we get (after redefining $k' \to k$)

$$\begin{aligned}\Lambda_\mu &= -2i(e\mu^\epsilon)^2 \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int \frac{d^n k}{(2\pi)^n} \frac{\gamma_\nu(\not p'(1-\beta) - \alpha\not p - \not k + m)\gamma_\mu(\not p(1-\alpha) - \beta\not p' - \not k + m)\gamma^\nu}{[k^2 - m^2(\alpha+\beta) + \alpha(1-\alpha)p^2 + \beta(1-\beta)p'^2 - \alpha\beta 2p'\cdot p]^3} \\ &\equiv \Lambda_\mu^{(1)} + \Lambda_\mu^{(2)}\,,\end{aligned} \tag{310}$$

where $\Lambda_\mu^{(1)}$ is the divergent piece where we keep the two $\not k$ terms in the numerator (the linear terms vanish by $k \to -k$ symmetry) and $\Lambda_\mu^{(2)}$ is the remaining convergent piece containing no k's in the numerator. Let us write

$$\gamma_\nu \not k \gamma_\mu \not k \gamma^\nu \;=\; k_\rho k_\sigma \gamma_\nu \gamma^\rho \gamma_\mu \gamma^\sigma \gamma^\nu \tag{311}$$

and then use Eq. (303) to write

$$\int \frac{d^n k}{(2\pi)^n} \frac{k_\rho k_\sigma}{[k^2 - m^2(\alpha+\beta) + \alpha(1-\alpha)p^2 + \beta(1-\beta)p'^2 - \alpha\beta 2p'\cdot p]^3}$$

$$
\begin{aligned}
&= i(-)^{n/2}\frac{1}{(4\pi)^{n/2}\Gamma(3)}\frac{1}{2}g_{\rho\sigma}\frac{\Gamma(2-\frac{n}{2})}{[-m^2(\alpha+\beta)+\alpha(1-\alpha)p^2+\beta(1-\beta)p'^2-\alpha\beta 2p'\cdot p]^{2-\frac{n}{2}}} \\
&= i\frac{1}{4(4\pi)^{n/2}}g_{\rho\sigma}\frac{\Gamma(\epsilon)}{[m^2(\alpha+\beta)-\alpha(1-\alpha)p^2-\beta(1-\beta)p'^2+\alpha\beta 2p'\cdot p]^{\epsilon}}\,. \qquad (312)
\end{aligned}
$$

Next, we use Eq. (293) to write

$$
\begin{aligned}
g_{\rho\sigma}\gamma_\nu\gamma^\rho\gamma_\mu\gamma^\sigma\gamma^\nu &= \gamma_\nu\gamma^\rho\gamma_\mu\gamma_\rho\gamma^\nu \\
&= \gamma_\nu(2-n)\gamma_\mu\gamma^\nu \\
&= (2-n)^2\gamma_\mu\,. \qquad (313)
\end{aligned}
$$

Putting all this into our expression for Λ_μ we get for just the coefficient of the singular $\frac{1}{\epsilon}$ part of $\Gamma(\epsilon)$ and setting $n=4$ ($\epsilon=0$) everywhere else:

$$
\begin{aligned}
\Lambda_\mu^{(1)} &= -2ie^2\int_0^1 d\alpha\int_0^{1-\alpha}d\beta\left[i\frac{1}{4(4\pi)^2\epsilon}\right]4\gamma_\mu + finite \\
&= \frac{e^2}{16\pi^2}\frac{1}{\epsilon}\gamma_\mu + finite\,. \qquad (314)
\end{aligned}
$$

For completeness, we could write out the finite part of $\Lambda_\mu^{(1)}$ as well and, of course, we have $\Lambda_\mu^{(2)}$ which is finite from the beginning. For the latter, we can simply set $n = 4$ and consider the non-k parts of the numerator. Using the integral of Eq. (301), we obtain

$$\Lambda_\mu^{(2)} = \frac{e^2}{16\pi^2}\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{\gamma_\nu(\not p'(1-\beta) - \alpha \not p + m)\gamma_\mu(\not p(1-\alpha) - \beta \not p' + m)\gamma^\nu}{[-m^2(\alpha+\beta) + \alpha(1-\alpha)p^2 + \beta(1-\beta)p'^2 - \alpha\beta 2p'\cdot p]}, \quad (315)$$

which we could compute explicitly if we really wanted to, as we should have to for precision comparisons with data, but will for the moment leave as it is. (Note: in this case I have an opposite overall sign wrp to Ryder. I think I am right, but please check.)

The 1-loop renormalization of QED

To summarize, we have found the singular structures:

$$\Sigma(p) = \frac{e^2}{16\pi^2\epsilon}(-\not p + 4m) + finite \quad (316)$$

$$\Pi_{\mu\nu}(k) = \frac{e^2}{12\pi^2\epsilon}(k_\mu k_\nu - k^2 g_{\mu\nu}) + finite \quad (317)$$

$$\Lambda_\mu^{(1)}(p,q,p') = \frac{e^2}{16\pi^2\epsilon}\gamma_\mu + finite\,. \quad (318)$$

Note that for the infinite terms above,

$$\Lambda_\mu = -\frac{\partial \Sigma(p)}{\partial p^\mu} . \tag{319}$$

This is one consequence of a very general result called the QED Ward identity.

An aside on the Ward identity

- This Ward identity holds to all orders in perturbation and it can be proved in a variety of ways. A path integral proof is given in Ryder, Sec. 7.4. Here, I will give a diagrammatic type of proof.

- The general statement of the Ward identity is

$$\frac{\partial S_F^{-1}(p)}{\partial p^\mu} = \Gamma_\mu(p, 0, p) , \tag{320}$$

  where S_F is the complete electron propagator and $e\Gamma_\mu$ is the complete photon-fermion-antifermion vertex function:

$$S_F(p)^{-1} = -i[\not{p} - m - \Sigma(p)] , \quad e\Gamma_\mu = -ie\gamma_\mu - ie\Lambda_\mu . \tag{321}$$

- The Ward identity is actually the $q^\mu \to 0$ limit of a more general identity called the Ward-Takahashi identity, which reads:

$$q^\mu \Gamma_\mu(p, q, p+q) = S_F^{-1}(p+q) - S_F^{-1}(p)\,. \tag{322}$$

To see how this comes about, let us imagine a diagram for $-i\Sigma(p)$. It will have an e line coming in and and e line exiting, and this e line will undergo various interactions at vertices within the diagram. There will also in general be internal closed e loops — we shall return to those in a moment. We establish a labeling for the vertices to which the photons internal to the diagram are attached, beginning with γ^{λ_0} at the entry point of the e line and ending with $\gamma^{\lambda_{n+1}}$ at the exit point of the e line. In between, we have vertices ranging from γ^{λ_1} to γ^{λ_n}. The fermion line running from the γ^{λ_0} vertex to the γ^{λ_1} vertex will have momentum p_1, and so forth with the final fermion line from γ^{λ_n} to $\gamma^{\lambda_{n+1}}$ having p_n.

We can generate a set of diagrams for the (amputated) γee vertex from the above diagram contributing to the propagator by inserting the $\gamma(q)$ (q is taken as ingoing) at every possible location along the e line and shifting all the momenta of the internal e propagators subsequent to the insertion point up by the additional momentum q. Each such insertion will have the standard $-ie\gamma^\mu$ interaction. When we multiply by q_μ, as we wish to do when we want to get the WT identity, this becomes an insertion of $-ieq_\mu\gamma^\mu$.

One then notes the following simple identity (factors of the charge e are removed in defining Λ_μ and Γ_μ):

$$-iq_\mu\gamma^\mu = -i[(\not p_i + \not q - m) - (\not p_i - m)]\,, \tag{323}$$

where p_i is the momentum on the e line to which we attached the $\gamma(q)$. Now, on either side of this insertion, we have an e propagator. Including these, we have the structure

$$\frac{i}{\not p_i + \not q - m}(-i\not q)\frac{i}{\not p_i - m} = \left(\frac{i}{\not p_i - m} - \frac{i}{\not p_i + \not q - m}\right)\,. \tag{324}$$

Including still more neighboring vertices and propagators, we end up with an expression of the form

$$\ldots\left(\frac{i}{\not p_{i+1} + \not q - m}\right)\gamma^{\lambda_{i+1}}\left(\frac{i}{\not p_i - m} - \frac{i}{\not p_i + \not q - m}\right)\gamma^{\lambda_i}\left(\frac{i}{\not p_{i-1} - m}\right)\gamma^{\lambda_{i-1}}\ldots \tag{325}$$

Now consider what happens if we insert $-i\not q$ into line $i-1$. We generate a structure of the form

$$\cdots\left(\frac{i}{\not p_{i+1}+\not q-m}\right)\gamma^{\lambda_{i+1}}\left(\frac{i}{\not p_i+\not q-m}\right)\gamma^{\lambda_i}\left(\frac{i}{\not p_{i-1}-m}-\frac{i}{\not p_{i-1}+\not q-m}\right)\gamma^{\lambda_{i-1}}\cdots \tag{326}$$

Clearly the 1st term in the 2nd expression cancels the 2nd term in the 1st expression. This will continue to work in this way until we get to the very ends of the fermion line, leaving us with two uncanceled forms so that we end up with (still need to return to closed fermion loops that we can show don't contribute)

$$\begin{aligned}
&[q^\mu\Gamma_\mu(p,q,p+q)] = \\
&-\gamma^{\lambda_{n+1}}\left(\frac{i}{\not p_n+\not q-m}\right)\gamma^{\lambda_n}\cdots\left(\frac{i}{\not p_{i+1}+\not q-m}\right)\gamma^{\lambda_{i+1}}\left(\frac{i}{\not p_i+\not q-m}\right)\gamma^{\lambda_i}\left(\frac{i}{\not p_{i-1}+\not q-m}\right)\gamma^{\lambda_{i-1}}\cdots \\
&\qquad\times\gamma^{\lambda_1}\left(\frac{i}{\not p_1+\not q-m}\right)\gamma^{\lambda_0} \\
&+\gamma^{\lambda_{n+1}}\left(\frac{i}{\not p_n-m}\right)\gamma^{\lambda_n}\cdots\left(\frac{i}{\not p_{i+1}-m}\right)\gamma^{\lambda_{i+1}}\left(\frac{i}{\not p_i-m}\right)\gamma^{\lambda_i}\left(\frac{i}{\not p_{i-1}-m}\right)\gamma^{\lambda_{i-1}}\cdots\gamma^{\lambda_1}\left(\frac{i}{\not p_1-m}\right)\gamma^{\lambda_0} \\
&= -[-i\Sigma(p+q)]+[-i\Sigma(p)]
\end{aligned} \tag{327}$$

where the $-i\gamma^\mu$ part of $-i\not q$ is included in the Γ^μ as being the vertex to which the external photon attaches (remember that in our definitions we have removed the explicit charge part, e, of the coupling — the full amputated vertex is $e\Gamma_\mu$).

- **If we include the bare vertex part of Γ_μ (i.e. $\Gamma_\mu = -i\gamma_\mu$+higher order) and recall that**

$$S_F^{-1}(p) = -i[\not{p} - m - \Sigma(p)] \tag{328}$$

so that

$$S_F^{-1}(p+q) - S_F^{-1}(p) = -i\not{q} + i\Sigma(p+q) - i\Sigma(p)\,, \tag{329}$$

we obtain the net result,

$$q^\mu \Gamma_\mu(p, q, p+q) = S_F^{-1}(p+q) - S_F^{-1}(p)\,. \tag{330}$$

In the limit as $q \to 0$, we can write

$$S_F^{-1}(p+q) - S_F^{-1}(p) = q^\mu \frac{\partial S_F^{-1}}{\partial p^\mu}\,. \tag{331}$$

Inserting above, and matching coefficients of q^μ as $q^\mu \to 0$ yields the Ward identity:

$$\Gamma_\mu(p, 0, p) = \frac{\partial S_F^{-1}}{\partial p^\mu}\,. \tag{332}$$

It is always useful to remember this by simply referring to the zeroth order version of the above which says that

$$-i\gamma_\mu = \frac{\partial \left[\frac{(\not p - m)}{i}\right]}{\partial p^\mu}. \tag{333}$$

- Returning to the issue of closed loops, I hope the following is obvious.

  After inserting $-i \not q$ at every possible location on the closed loop, since the loop is closed one ends up with the closed loop written in terms of the unshifted p_i's that appeared in the loop *minus* exactly the same structure with every $p_i \rightarrow p_i + q$. In other words, we will have a result proportional to

$$\mathrm{Tr}\left\{ -\left(\frac{i}{\not p_n + \not q - m}\right)\gamma^{\lambda_n}\ldots\left(\frac{i}{\not p_{i+1} + \not q - m}\right)\gamma^{\lambda_{i+1}}\left(\frac{i}{\not p_i + \not q - m}\right)\gamma^{\lambda_i}\left(\frac{i}{\not p_{i-1} + \not q - m}\right)\gamma^{\lambda_{i-1}}\ldots\left(\frac{i}{\not p_1 + \not q - m}\right)\right.$$

$$\left. + \left(\frac{i}{\not p_n - m}\right)\gamma^{\lambda_n}\ldots\left(\frac{i}{\not p_{i+1} - m}\right)\gamma^{\lambda_{i+1}}\left(\frac{i}{\not p_i - m}\right)\gamma^{\lambda_i}\left(\frac{i}{\not p_{i-1} - m}\right)\gamma^{\lambda_{i-1}}\ldots\left(\frac{i}{\not p_1 - m}\right)\right\}. \tag{334}$$

But, every one of these p_i's is given in terms of the loop integration variable, lets use p_1 as the free loop momentum, plus momenta that are brought in at the vertices that are present on the closed loop. These "external" momenta can be regarded as fixed as we perform the p_1 loop integration. But, then the term with all the p_i replaced by $p_i + q$ can be converted to the same form as the term without any q's added simply by shifting the integration variable, $p_1 + q \to p_1$, which will also shift all the other $p_i + q \to p_i$. Because there was a relative minus sign between the $p_i + q$ and the p_i structures, they cancel after this shift.

It should be noted that this shift is not always valid. Theories such as QED for which this shift *is* valid are called "non-anomalous". If the theory has an anomaly, this shift is not valid. Anomalies can emerge when considering axial vector currents such as those encountered in the electroweak interactions. We will not have time to pursue this topic, but you will want to read about it in one of the standard texts (e.g. Peskin section 19.2). In the context of dimensional regularization, the problem is associated with defining γ_5 in $n \neq 4$ dimensions. One finds (see Peskin section 19.4) that the electroweak theory will be anomalous (certain unphysical states would fail to cancel and the S matrix would not be unitary) unless quarks and leptons come in just the right representations. The condition that the electroweak theory be

anomaly free is $\mathrm{Tr}[Q] = 0$, where Q is the charge operator and $\mathrm{Tr}[Q]$ is the sum over all the charges of the fermions of the theory. For one family of quarks and leptons we have

$$\mathrm{Tr}[Q] = 3\left(\frac{2}{3} - \frac{1}{3}\right) + (0 - 1) = 0\,, \tag{335}$$

where the first term comes from the quarks with three colors and the second term comes from the leptons. Thus, if quarks have fractional charge, they must have three colors for this to work. The structure of each family and the need for complete families is no accident!

Back to the renormalization program

The Ward identity is very critical to proving renormalizability to all orders. As we have seen, it is already incorporated in our explicit one-loop results.

So let us now construct the counter-term Lagrangian, $\Delta\mathcal{L}$.

- We begin with an appropriate term to counter the singularity in $-i\Sigma(p)$. There are actually two independent singularities:

$$-i\Sigma(p) = -i\frac{e^2}{16\pi^2\epsilon}(-\not{p} + 4m) + finite \tag{336}$$

which must be countered by terms of the form

$$\Delta\mathcal{L} = iB\overline{\psi}\,\partial\!\!\!/\psi - A\overline{\psi}\psi\,. \tag{337}$$

The Feynman rules that go with these $\Delta\mathcal{L}$ terms are simply $iB\,p\!\!\!/$ and $-iA$, respectively. (We always have to multiply by i from the $\exp[i\Delta\mathcal{L}]$ expansion to first order, and $\partial\!\!\!/ \rightarrow -i\,p\!\!\!/$ in momentum space.) What we need is for

$$-i\Sigma(p) - iA + iB\,p\!\!\!/ = -i\frac{e^2}{16\pi^2\epsilon}(-\,p\!\!\!/ + 4m) + finite - iA + iB\,p\!\!\!/ = finite. \tag{338}$$

This is the case provided

$$A = -\frac{me^2}{4\pi^2\epsilon}, \quad B = -\frac{e^2}{16\pi^2\epsilon}\,. \tag{339}$$

So, if we match $\mathcal{L} + \Delta\mathcal{L}$ to $\mathcal{L}_0$ for the purely fermionic part, we must have

$$i(1+B)\overline{\psi}\,\partial\!\!\!/\psi - (m+A)\overline{\psi}\psi = i\overline{\psi}_0\,\partial\!\!\!/\psi_0 - m_0\overline{\psi}_0\psi_0\,. \tag{340}$$

As we know, this is accomplished by wave function and mass renormalization in the form

$$\psi_0 = \sqrt{Z_2}\psi\,, \quad m_0 = m + \delta m\,, \tag{341}$$

where the convention is to call the QED wave function renormalization factor Z_2. From this we see that

$$Z_2 = 1 + B = 1 - \frac{e^2}{16\pi^2\epsilon}\,,$$
$$m_0 = Z_2^{-1}(m + A) = m\left(1 - \frac{e^2}{4\pi^2\epsilon}\right)\left(1 + \frac{e^2}{16\pi^2\epsilon}\right) = m\left(1 - \frac{3e^2}{16\pi^2\epsilon}\right) = m + \delta m \tag{342}$$

To repeat what we have already considered in the ϕ^4 theory case, we have the relation

$$\langle 0|T\{\overline{\psi}_0(x)\psi_0(y)\}|0\rangle = Z_2\langle 0|T\{\overline{\psi}(x)\psi(y)\}|0\rangle \tag{343}$$

which, in words, says that $Z_2 < 1$ is the probability of finding a full renormalized electron (represented by ψ) propagating in the propagation of a bare electron (represented by ψ_0). Alternatively, we say that in order to get full one-particle normalization for ψ propagation, we must boost up ψ_0 propagation by $Z_2^{-1} > 1$.

There is one more important feature of the result for δm. Note that $\delta m \to 0$ as $m \to 0$. This is a consequence of chiral symmetry. You should recall that the Dirac $\mathcal{L}$ has an extra symmetry under $e^{-i\alpha\gamma_5}$ chiral transformations in the $m = 0$ limit. This symmetry is retained by the full QED Lagrangian (keeping $m = 0$). Thus, this symmetry should be respected to all orders of perturbation theory. In order for this to be the case, a non-zero value for the electron mass should not be generated by loop corrections. We see that our explicit 1-loop result verifies this requirement.

This behavior means that the chiral symmetry has "protected" the electron mass from hierarchy/fine-tuning problems. To understand what we mean by this, let us compare to the ϕ^4 theory. There, the one-loop correction to the scalar propagator gives rise to a quadratically divergent correction to the scalar particle mass-squared. The net result is that

$$m^2 = m_0^2 + \lambda\Lambda^2 \tag{344}$$

where Λ is the cutoff of the loop momentum integration. If the observed m is of modest observable size (e.g. of order a TeV), then if Λ is very large (for example of order $M_{\rm P} \sim 10^{19}~{\rm GeV}$) this modest size becomes unnatural in the sense that a very precise cancellation (to a few parts in 10^{38}) between m_0^2 and $\lambda\Lambda^2$ would be required. This is termed the "fine-tuning" problem that derives from the "hierarchy" problem of understanding how two such

diverse scales as $M_{\rm P}$ and $1~{\rm TeV}$ can be accommodated in the theory. Of course, this problem for the scalar field is why we have come to favor a new symmetry, such as supersymmetry, in which there are, for example, spin-1/2 supersymmetric partners of mass $\sim m_{\rm SUSY}$ to the scalar particles that also give quadratically divergent contributions to m^2. However, these quadratic divergences come with the opposite sign of the fermion loop and the supersymmetry also implies that the couplings involved are of exactly the correct size such that once the loop momentum l exceeds $m_{\rm SUSY} \sim 1~{\rm TeV}$ there is exact cancellation of the scalar loop correction against the fermion loop correction. In this case, Λ is replaced by $m_{\rm SUSY}$ and if $m_{\rm SUSY} \sim 1~{\rm TeV}$ there is no fine-tuning problem even if the ultimate ultraviolet completion of the theory (but now including SUSY) is at $M_{\rm P}$.

In the electron/QED case, the above kind of problem and the necessity of some such dramatic solution does not arise. Whatever the mass, m, of the electron is, the 1-loop-corrected mass has the form $m(1 + \frac{\alpha}{4\pi} \ln \frac{\Lambda}{\mu})$, where I have replaced the $\frac{1}{\epsilon}$ by the momentum cutoff equivalent. Now, even if $\Lambda \sim M_{\rm P}$ the correction *factor* is not particularly big ($\sim 1 + \frac{1}{516\pi} \times 19$) and so even if it is gravity that provides the ultraviolet completion of the theory there is no particular fine-tuning problem.

- **Now let us turn to the vacuum polarization correction to the photon propagator and see what role it plays.**

  First, we must figure out what the counter term Lagrangian $\Delta\mathcal{L}$ must be for this part of things. We are dealing with the

$$\mathcal{L} \ni -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 = \frac{1}{2}A^\mu g_{\mu\nu}\Box A^\nu \tag{345}$$

  part of $\mathcal{L}$. The allowed counter terms are then of the form

$$\Delta\mathcal{L} = -\frac{C}{4}F_{\mu\nu}F^{\mu\nu} - \frac{E}{2}(\partial_\mu A^\mu)^2 \,. \tag{346}$$

  Note that C and E are not necessarily equal in the counter term, even though they were for $\mathcal{L}$. It is simply a matter of what is required and it turns out that $C \neq E$ is required. Now, the Feynman rules coming from the above $\Delta\mathcal{L}$ are:

$$\begin{aligned} &-iC(k^2 g_{\mu\nu} - k_\mu k_\nu)\,, \\ &-iEk^2 g_{\mu\nu}\,. \end{aligned} \tag{347}$$

Meanwhile our singularity structure is purely of the form

$$i\Pi_{\mu\nu}(k) = i\frac{e^2}{12\pi^2\epsilon}(k_\mu k_\nu - k^2 g_{\mu\nu}) = -i\frac{e^2}{12\pi^2\epsilon}(k^2 g_{\mu\nu} - k_\mu k_\nu) \,. \quad (348)$$

From this, we see that, as far as singularities are concerned we may take $E = 0$. This is important since it means we can remain in Feynman gauge for which the precise $\mathcal{L} \ni -\frac{1}{2}(\partial_\mu A^\mu)^2$ form and normalization is required. Thus, subsequent calculations in still higher order can again be done in Feynman gauge. This is not to say that the resummed photon propagator will have a Feynman gauge form, but that is a different issue as we shall see.

Anyway, the above requires that

$$C = -\frac{e^2}{12\pi^2\epsilon} \quad (349)$$

at least as far as the singular part is concerned.

Before continuing on, let me return to a schematic derivation of the claim that the Feynman rule for the C term of $\Delta\mathcal{L}$ is indeed as stated above in

Eq. (347). We begin by using the antisymmetry to write

$$\mathcal{L}_{FF} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu) \,. \quad (350)$$

In order to determine the corresponding Feynman rule, we must compute the S matrix for a photon to turn back into a photon after interacting with $\mathcal{L}_{FF}$. The Feynman rule for $i\mathcal{M}$ will be extracted from the S form by "pulling away" the usual factors (which you will see shortly). We thus have to compute, leaving aside the C coefficient for the moment,

$$\langle 0| a^s_{\vec k} \left[i \int d^4x \mathcal{L}_{FF} \right] a^{t\,\dagger}_{\vec k} |0\rangle \quad = \quad \langle 0| a^s_{\vec k} \int d^4x \left[-\frac{i}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu) \right] a^{t\,\dagger}_{\vec k} |0\rangle \quad (351)$$

where, as always, the decomposition of the A_μ fields looks like

$$A_\mu(x) = \sum_{\vec q, r} \frac{1}{\sqrt{2VE_{\vec q}}} \left(a^r_{\vec q} \epsilon^r_\mu(\vec q) e^{-iq\cdot x} + a^{r\,\dagger}_{\vec q} \epsilon^r_\mu(\vec q) e^{iq\cdot x} \right) \quad (352)$$

assuming a real ϵ^r polarization basis. Obviously we must introduce two such expansions. We write the 2nd expansion as a $\sum_{\vec p, r'}$ expansion. Let us focus first on just one of the two equivalent contraction matchings, or "killing" operations. We contract the $a^{r'}_{\vec p}$ with the $a^{t\,\dagger}_{\vec k}$ and the $a^s_{\vec k}$ with the $a^{r\,\dagger}_{\vec{q}}$, yielding $\delta_{r't}\delta_{\vec{p}\vec{k}}\delta_{rs}\delta_{\vec{q}\vec{k}}$. We write the derivatives in terms of the momenta q and p appearing in the expansions. However, after employing the above δ's, we have $p = q = k$. The result is:

$$\begin{aligned}
&\langle 0|a^{s}_{\vec{k}} \int d^4x \left[-\frac{i}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu) \right] a^{t\,\dagger}_{\vec{k}}|0\rangle_{\text{contraction 1}} \\
= \quad & -\frac{i}{2}\frac{1}{\sqrt{2VE_{\vec{k}}}\sqrt{2VE_{\vec{k}}}} \int d^4x e^{-ik\cdot x+ik\cdot x} \left\{ \epsilon^{s}_{\nu}(-ik_\mu)(+ik^\mu)\epsilon^{t\,\nu} - \epsilon^{s}_{\mu}(-ik_\nu)(+ik^\mu)\epsilon^{t\,\nu} \right\} \\
= \quad & \epsilon^{s\,\mu}\epsilon^{t\,\nu}\frac{1}{\sqrt{2VE_{\vec{k}}}\sqrt{2VE_{\vec{k}}}}(2\pi)^4\delta^4(k-k)\left(\frac{-i}{2}\right)\left[k^2 g_{\mu\nu} - k_\mu k_\nu\right] . \qquad (353)
\end{aligned}$$

The other possible contraction arrangement doubles this answer. Removing the usual factors in extracting $i\mathcal{M}$, we get the Feynman rule for $i\mathcal{M}$ of

$$-\,i\left[k^2 g_{\mu\nu} - k_\mu k_\nu\right] \qquad (354)$$

as claimed.

So, the net form of the pure gauge part of the Lagrangian is:

$$\mathcal{L} + \Delta\mathcal{L} \ni -\left(\frac{1+C}{4}\right) F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 \equiv -\frac{Z_3}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 \,, \tag{355}$$

where at 1-loop we have

$$Z_3 = 1 + C = 1 - \frac{e^2}{12\pi^2\epsilon} \,. \tag{356}$$

The above pure gauge part of $\mathcal{L} + \Delta\mathcal{L}$ is to be identified with the bare Lagrangian form

$$\mathcal{L}_0 = -\tfrac{1}{4}F_{0\,\mu\nu}F_0^{\mu\nu} + \text{gauge terms} \,, \tag{357}$$

where $F_0^{\mu\nu} = \partial^\mu A_0^\nu - \partial^\nu A_0^\mu$. This means that we should relate the unrenormalized ("bare") gauge field to the renormalized gauge field by:

$$A_0^\mu = \sqrt{Z_3} A^\mu \,. \tag{358}$$

We would have a relation very much like that for the fermion field:

$$\langle 0|T\{A_0^\mu(x)A_0^\nu(y)\}|0\rangle \sim Z_3\langle 0|T\{A^\mu(x)A^\nu(y)\}|0\rangle \tag{359}$$

giving rise to the interpretation that Z_3 is the probability that the bare field A_0^μ will propagate freely like a non-interacting particle.

In any case, we have seen that we can choose C so that the sum of the actual vacuum polarization diagram $i\Pi_{\mu\nu}$ plus the counter term leaves us with a form

$$i\widetilde{\Pi}_{\mu\nu} = i\Pi(k^2)\left[k^2 g_{\mu\nu} - k_\mu k_\nu\right] \equiv i\Pi(k^2)P_{\mu\nu}(k)\,, \tag{360}$$

where $\Pi(k^2)$ is finite, and depends upon the scheme (i.e. on the finite terms in addition to the $\frac{1}{\epsilon}$ that we put into C.

Now a crucial identity:

$$\begin{aligned} P_{\alpha\beta}(k)\frac{g^{\beta\lambda}}{k^2}P_{\lambda\mu} &= P_{\alpha\lambda}P^\lambda_\mu\frac{1}{k^2} \\ &= (k^2 g_{\alpha\lambda} - k_\alpha k_\lambda)(k^2 g^\lambda_\mu - k^\lambda k_\mu)\frac{1}{k^2} \end{aligned}$$

$$
\begin{aligned}
&= (k^4 g_{\alpha\mu} - 2k_\alpha k_\mu k^2 + k_\alpha k_\mu k^2)\frac{1}{k^2} \\
&= P_{\alpha\mu}(k) \,. \qquad (361)
\end{aligned}
$$

With this in hand, we can iterate this 1PI irreducible object to develop a result for the full propagator:

$$
\begin{aligned}
D_{\rho\sigma}(k) &= \frac{-ig_{\rho\sigma}}{k^2} + \frac{-ig_{\rho\alpha}}{k^2} i\widetilde{\Pi}_{\alpha\beta}(k^2)\frac{-ig_{\beta\sigma}}{k^2} \\
&\quad + \frac{-ig_{\rho\alpha}}{k^2} i\widetilde{\Pi}_{\alpha\beta}(k^2)\frac{-ig_{\beta\lambda}}{k^2} i\widetilde{\Pi}_{\lambda\nu}(k^)\frac{-ig_{\nu\sigma}}{k^2} + \ldots \\
&= \frac{-ig_{\rho\sigma}}{k^2} + \frac{-i(k^2 g_{\rho\sigma} - k_\rho k_\sigma)}{k^4}\left[\Pi(k^2) + \Pi^2(k^2) + \ldots\right] \\
&= \frac{-ig_{\rho\sigma}}{k^2} + \frac{-i(k^2 g_{\rho\sigma} - k_\rho k_\sigma)}{k^4}\frac{\Pi(k^2)}{1 - \Pi(k^2)} \\
&= \frac{-ig_{\rho\sigma}}{k^2}\frac{1}{1 - \Pi(k^2)} + \frac{ik_\rho k_\sigma}{k^4}\frac{\Pi(k^2)}{1 - \Pi(k^2)} \,. \qquad (362)
\end{aligned}
$$

We can basically ignore the 2nd "gauge" term of this expression; it will not contribute to any physical amplitude because of the $k^\mu \mathcal{M}_\mu = 0$ type of identity (which is actually a consequence of the Ward-Takahashi identity). From this, we see that so long as $\Pi(k^2)$ does not have a singular behavior as $k^2 \to 0$ (e.g. $\Pi(k^2) = \frac{a}{k^2}$), which it does not by explicit calculation, the full photon propagator will still have a pole at $k^2 = 0$ implying that renormalization has not ended up giving a mass to the photon. We see that this is intimately related to the gauge invariant structure, namely $\widetilde{\Pi}_{\mu\nu}(k) \propto P_{\mu\nu}(k)$. Note that had there been such a singular behavior for $\Pi(k^2)$, the full photon propagator would behave as $\frac{1}{k^2 - a}$, corresponding (roughly, i.e. modulo whatever other finite terms are present for $k^2 \to a$) to a photon mass-squared $= a$.

- Finally, we have the vertex correction to deal with.

  To cancel the infinity in $\Lambda_\mu^{(1)}$ we must introduce a counter term of the form:

$$\Delta \mathcal{L} \ni -De\mu^\epsilon \overline{\psi} \not{A} \psi \tag{363}$$

  which in the standard fashion will give rise to a Feynman rule of form:

$$-iDe\mu^\epsilon \gamma_\mu \,. \tag{364}$$

D must be chosen, therefore, so that

$$-ie\mu^{\epsilon}(\Lambda_{\mu}^{(1)}+D\gamma_{\mu}) = finite\,, \quad \text{with} \quad \Lambda_{\mu}^{(1)} = \frac{e^2}{16\pi^2\epsilon}\gamma_{\mu}+finite\,. \tag{365}$$

We make the simple choice of

$$D = -\frac{e^2}{16\pi^2\epsilon}\,. \tag{366}$$

The full structure is then:

$$\begin{aligned} \mathcal{L}+\Delta\mathcal{L} &\ni -(1+D)e\mu^{\epsilon}A^{\mu}\overline{\psi}\gamma_{\mu}\psi \\ &\equiv -Z_1 e\mu^{\epsilon}A^{\mu}\overline{\psi}\gamma_{\mu}\psi\,, \end{aligned} \tag{367}$$

with

$$Z_1 = 1+D = 1-\frac{e^2}{16\pi^2\epsilon}\,. \tag{368}$$

Thus, in the end we have the three multiplicative renormalization factors:

$$Z_1 = Z_2 = 1-\frac{e^2}{16\pi^2\epsilon}\,, \quad Z_3 = 1-\frac{e^2}{12\pi^2\epsilon}\,. \tag{369}$$

And, you should of course realize that $Z_1 = Z_2$ is no pure coincidence; it is required by the Ward identity as we have already stressed.

Finally, we can work out the relation between the bare charge, e_0, and the renormalized charge, e. We must have

$$\mathcal{L} + \Delta\mathcal{L} = \mathcal{L}_0 \,, \tag{370}$$

which for the photon fermion fermion interaction term leads to the requirement

$$-Z_1 e\mu^\epsilon A^\mu \overline{\psi}\gamma_\mu\psi \quad = \quad -e_0 A_0^\mu \overline{\psi}_0 \gamma_\mu \psi_0 = -e_0\sqrt{Z_3}(\sqrt{Z_2})^2 A^\mu \overline{\psi}\gamma_\mu\psi \,, \tag{371}$$

where we inserted the relations of the bare fields to the renormalized fields established earlier. This implies the relation

$$e_0 = e\mu^\epsilon \frac{Z_1}{Z_2 Z_3^{1/2}} = e\mu^\epsilon Z_3^{-1/2} \,, \tag{372}$$

where the latter equality follows from the Ward identity requirement (and also our explicit 1-loop result) of $Z_1 = Z_2$.

- **Summary**

We have absorbed all the infinite quantities into the definitions of the bare fields and bare mass. The fact that we were able to do so, keeping a $\mathcal{L}$ of the same form as the original means that, to this 1-loop order, QED is indeed renormalizable. The proof that the renormalization procedure can be carried out to all orders appears in Ryder and other books.

Here, I wish to focus on some more implications of the 1-loop results.

- The asymptotic behavior or $e(\mu)$.

We begin with the result of Eq. (372):

$$\begin{aligned} e_0 &= e\mu^\epsilon Z_3^{-1/2} \\ &= e\mu^\epsilon \left(1 - \frac{e^2}{12\pi^2\epsilon}\right)^{-1/2} \\ &= e\mu^\epsilon \left(1 + \frac{e^2}{24\pi^2\epsilon}\right) + \mathcal{O}(e^4)\,. \end{aligned} \tag{373}$$

We now differentiate in the usual fashion:

$$0 = \mu \frac{\partial e_0}{\partial \mu} = \epsilon e \mu^\epsilon \left(1 + \frac{e^2}{24\pi^2 \epsilon}\right) + \mu \frac{\partial e}{\partial \mu} \left[\mu^\epsilon \left(1 + \frac{e^2}{24\pi^2\epsilon}\right) + e\mu^\epsilon \frac{e}{12\pi^2\epsilon}\right]$$
$$= \epsilon e \mu^\epsilon \left(1 + \frac{e^2}{24\pi^2 \epsilon}\right) + \mu \frac{\partial e}{\partial \mu} \mu^\epsilon \left[1 + \frac{e^2}{8\pi^2\epsilon}\right] . \quad (374)$$

Solving for $\mu \frac{\partial e}{\partial \mu}$ gives

$$\mu \frac{\partial e}{\partial \mu} = -\epsilon e \frac{1 + \frac{e^2}{24\pi^2\epsilon}}{1 + \frac{e^2}{8\pi^2\epsilon}}$$
$$= -\epsilon e \left(1 - \frac{e^2}{12\pi^2\epsilon} + \mathcal{O}(e^4)\right)$$
$$\stackrel{\epsilon \to 0}{\longrightarrow} \frac{e^3}{12\pi^2}$$
$$\equiv \beta(e) . \quad (375)$$

Thus, as in ϕ^4 theory, $\beta(e) > 0$. The solution of the above equation, which is obvious after rewriting it in the form:

$$\mu\frac{\partial e^2}{\partial\mu} = \frac{(e^2)^2}{6\pi^2} \tag{376}$$

is

$$e^2(\mu) = \frac{e^2(\mu_0)}{1 - \frac{e^2(\mu_0)}{6\pi^2}\ln\frac{\mu}{\mu_0}} \tag{377}$$

and we see that $e^2(\mu)$ increases with increasing μ or decreasing distance scale. The singular point at

$$\mu = \mu_0 \exp\left(\frac{6\pi^2}{e^2(\mu_0)}\right) \tag{378}$$

is sometimes referred to as the *Landau singularity*.

This increase of $e(\mu)$ at short distances has the following interpretation. We begin at large distances and imagine the photon probing an electron, e^-. At large distances, the photon has lots of sequential fermion loop (bubble) insertions of e^+e^- pairs in its propagator before it interacts with the electron. These bubbles polarize (like a dielectric medium) so that the

e^+ of each bubble insertion spends more time near the source e^- than does the e^- of each bubble. Thus, the fundamental charge of the source e^- is substantially shielded by the "polarization of the vacuum" (that is where the name comes from). As we probe on shorter and shorter distance scales (higher momentum scales), we penetrate further and further inside this cloud of bubbles and we see more and more of the bare e^- charge.

- The anomalous magnetic moment of the electron.

For this we will need to return to $\Lambda_\mu^{(2)}$. To set the stage, we first derive the standard tree-level result that $g_e = 2$, the so-called Dirac magnetic moment for the electron.

Before proceeding, we need to prove the Gordon identity. One begins with the vertex γ_μ. This we sandwich between spinors in the form $\overline{u}(p')\gamma_\mu u(p)$ to establish the interaction of an electron with the photon at tree-level. Recalling that

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}\,, \quad \gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu = -2i\sigma_{\mu\nu} \tag{379}$$

we can write, using the Dirac equation for the spinors,

$$\begin{aligned}\overline{u}(p')\gamma_\mu u(p) &= \frac{1}{2m}\overline{u}(p')(\not{p}'\gamma_\mu + \gamma_\mu \not{p})u(p) \\ &= \frac{1}{2m}\overline{u}(p')(p'^{\,\nu}\gamma_\nu\gamma_\mu + \gamma_\mu\gamma_\nu p^\nu)u(p) \\ &= \frac{1}{2m}\overline{u}(p')(p'^{\,\nu}[g_{\mu\nu} + i\sigma_{\mu\nu}] + [g_{\mu\nu} - i\sigma_{\mu\nu}]p^\nu)u(p) \\ &= \frac{1}{2m}\overline{u}(p')[(p'_\mu + p_\mu) + i\sigma_{\mu\nu}q^\nu]u(p)\,. \end{aligned} \tag{380}$$

This is the so-called Gordon identity.

To use this identity, we first consider the full vertex Γ_μ. Given the 4-vectors available, the most general possible expression is

$$\Gamma_\mu(p, q, p' = p + q) = \gamma_\mu A + (p' + p)_\mu B + (p' - p)_\mu C\,. \tag{381}$$

We have already seen that the other possible form, $\sigma_{\mu\nu}q^\nu$, is equivalent to a linear combination of the first two terms above when the structure is sandwiched between spinors. Further, the coefficients A, B, and C could all contain Dirac matrices contracted with vectors, *i.e.* $\not{p}$, or $\not{p}'$. But, if we plan to sandwich Γ_μ between spinors, $\overline{u}(p')\Gamma_\mu u(p)$, then $\not{p} \to m$ and $\not{p}' \to m$ and so there is for our purposes no need to consider A, B, C as anything other than simple numbers. Given that we will also require $p^2 = p'^2 = m^2$, the only non-trivial number is $q^2 = -2p \cdot p' + 2m^2$. A, B, C can be functions of q^2. However, we can simplify even further. Gauge invariance for the on-shell external in and out electron situation being considered requires that $q^\mu \overline{u}(p')\Gamma_\mu u(p) = 0$. This follows from the Dirac equation for the γ_μ term, and from $(p'-p)\cdot(p'+p) = p'^2 - p^2 = m^2 - m^2$ for the 2nd term, but is not true for the 3rd term. Thus, $C = 0$ is required.

At this point, it is conventional to replace the $(p'+p)_\mu$ term using the Gordon identity, and so we can say that the most general structure for the vertex is

$$\overline{u}(p')\Gamma_\mu u(p) = \overline{u}(p')\left[\gamma_\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2m} F_2(q^2)\right] u(p) \,. \qquad (382)$$

F_1 and F_2 are called form factors. The hall-mark of an elementary particle is that at tree-level $F_1 = 1$ and $F_2 = 0$.

To see how this structure is related to the magnetic moment of the electron, we must allow the electron to interact with a classical vector potential,

$A^{\mu}_{cl}(x) = (0, \vec{A}_{cl}(\vec{x}))$. It is the Fourier transform of this potential that appears in the $i\mathcal{M}$ matrix element for the electron scattering from this classical potential. To see exactly how, you consider a

$$\Delta H_{int} = \int d^3\vec{x} e A^{\mu}_{cl} j_{\mu} \,, \quad \text{with} \quad j_{\mu}(x) = \overline{\psi}(x)\Gamma_{\mu}\psi(x) \,. \tag{383}$$

In the leading order of perturbation theory, the S-matrix element for scattering from this classical field is[1]

$$i\mathcal{M}(2\pi)\delta(p^{0\prime} - p^0) = -ie\overline{u}(p')\Gamma_{\mu}u(p)\widehat{A}^{\mu}_{cl}(p' - p) \tag{384}$$

where $\widehat{A}^{\mu}_{cl}(p' - p)$ is the 4-dimensional Fourier transform of $A^{\mu}_{cl}(x)$ which for a time-independent potential always takes the form

$$\widehat{A}^{\mu}_{cl}(p' - p) = (2\pi)\delta(p^{0\prime} - p^0)\widetilde{A}^{\mu}_{cl}(\vec{q}) \tag{385}$$

[1] The $(2\pi)^3\delta^3(\vec{p}' - \vec{p})$ is missing since 3-momentum conservation does not apply when dealing with scattering from a classical potential which is "infinitely massive" and can absorb any amount of momentum. To get the expression given, one simply considers the 1st order term in the expansion of $\exp[-i\int dt \Delta H_{int}]$ sandwiched between initial and final states, $\langle p'| - i\int dt \Delta H_{int}|p\rangle$, does the 1-particle contractions, and removes the usual $1/\sqrt{2VE}$ factors for the initial and final particles.

where $\widetilde{A}^{\mu}_{cl}(\vec{q})$ is the 3-dimensional Fourier transform. The result is that

$$i\mathcal{M} = -ie\overline{u}(p')\Gamma_\mu u(p)\widetilde{A}^{\mu}_{cl}(\vec{q})\,. \tag{386}$$

In the case of $A^{\mu}_{cl}(x) = (0, \vec{A}_{cl}(\vec{x}))$, as appropriate for a magnetic field, the above expression for $i\mathcal{M}$ takes the form[2]

$$i\mathcal{M} = ie\widetilde{A}^{i}_{cl}(\vec{q})\overline{u}(p')\left[\gamma^i F_1(q^2) + \frac{i\sigma^{i\nu}q_\nu}{2m}F_2(q^2)\right]u(p)\,. \tag{387}$$

The coefficient of $\widetilde{A}_{cl}$ vanishes as $q \to 0$ and so we must be careful to extract the linear term in q. We need the nonrelativistic expansion of the spinors $u(p)$ and $u(p')$. The relevant expansion is

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\overline{\sigma}}\,\xi \end{pmatrix} \simeq \sqrt{m}\begin{pmatrix} (1 - \frac{\vec{p}\cdot\vec{\sigma}}{2m})\xi \\ (1 + \frac{\vec{p}\cdot\vec{\sigma}}{2m})\xi \end{pmatrix}\,. \tag{388}$$

[2]The usual $-$ sign has been compensated by my having 2 up i indices rather than one up and one down. This is just the first of many such little sign changes associated with raised or lowered spatial indices. You have to be careful to get them all in correctly.

Then, the F_1 term can be simplified in the form:

$$\overline{u}(p')\gamma^i u(p) = 2m\xi'^\dagger \left(\frac{\vec{p}' \cdot \vec{\sigma}}{2m}\sigma^i + \sigma^i \frac{\vec{p} \cdot \vec{\sigma}}{2m} \right) \xi \,. \tag{389}$$

we then use $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$ to obtain a spin-independent term proportional to $\vec{p}' + \vec{p}$ (from the δ^{ij} part) and a spin-dependent term proportional to $\vec{p}' - \vec{p}$ (from the $\epsilon^{ijk}\sigma^k$ part). The first of these terms is the contribution of the operator $[\vec{p}_{op} \cdot \vec{A} + \vec{A} \cdot \vec{p}_{op}]$ in the standard kinetic energy term of nonrelativistic quantum mechanics. The 2nd term is the magnetic moment interaction we are after, and takes the form

$$\overline{u}(p')\gamma^i u(p) = 2m\xi'^\dagger \left(\frac{-i}{2m}\epsilon^{ijk} q^j \sigma^k \right) \xi \,. \tag{390}$$

The F_2 term already contains an explicit factor of q, so we can evaluate it using the leading term in the spinor expansion to obtain (sign change since we raise the index on q_j)

$$\overline{u}(p') \left(\frac{i}{2m}\sigma^{i\nu} q_\nu \right) u(p) = 2m\xi'^\dagger \left(\frac{-i}{2m}\epsilon^{ijk} q^j \sigma^k \right) \xi \,. \tag{391}$$

Thus, the full spin-dependent structure takes the form

$$\overline{u}(p')\left(\gamma^i F_1 + \frac{i\sigma^{i\nu}q_\nu}{2m}F_2\right)u(p) \overset{q\to 0}{\to} 2m\xi'^{\dagger}\left(\frac{-i}{2m}\epsilon^{ijk}q^j\sigma^k\left[F_1(0)+F_2(0)\right]\right)\xi\,, \tag{392}$$

which gives

$$i\mathcal{M} = -i(2m)e\xi'^{\dagger}\left(\frac{-1}{2m}\sigma^k\left[F_1(0)+F_2(0)\right]\right)\xi\widetilde{B}^k(\vec{q})\,, \tag{393}$$

where

$$\widetilde{B}^k(\vec{q}) = -i\epsilon^{ijk}q^i\widetilde{A}^j_{cl}(\vec{q}) \tag{394}$$

is the Fourier transform of the magnetic field $\vec{B} = \vec{\nabla}\times\vec{A}$ produced by $\vec{A}_{cl}(\vec{x})$. We interpret this expression as the Born approximation to the scattering of the electron from a potential. The potential takes the form[3]

[3] The $2m$ factor in (393) is part of our relativistic normalization and should be dropped in comparing to the usual Born approximation — see Peskin Eq. (4.123) and surrounding discussion. The general rule in relativistic normalization is that

$$i\mathcal{M} = -i2m\widetilde{V}_{cl}(\vec{q})\,, \tag{395}$$

where $\widetilde{V}_{cl}(\vec{q})$ is the 3-dimensional Fourier transform of the effective spatial potential $V_{cl}(\vec{x})$.

of a magnetic moment interaction

$$V(\vec{x}) = -\langle \vec{\mu} \rangle \cdot \vec{B}(\vec{x}) \tag{396}$$

with the identification of

$$\langle \vec{\mu} \rangle = \frac{e}{m} \left[F_1(0) + F_2(0) \right] \xi'^{\dagger} \frac{\vec{\sigma}}{2} \xi \,. \tag{397}$$

This should be compared to the standard form of

$$\vec{\mu} = g \left(\frac{e}{2m} \right) \vec{S} \tag{398}$$

where $\vec{S}$ is the electron spin operator $\frac{\vec{\sigma}}{2}$. Doing so, we find

$$g = 2 \left[F_1(0) + F_2(0) \right] = 2 + 2F_2(0) = 2 + \mathcal{O}(\alpha) \,, \tag{399}$$

where g is called the *Lande g-factor*.

You might ask how I can write so confidently that $F_1(0) = 1$ without doing the loop calculation. Well, it is because $F_1(0)$ *defines* the charge! To see this, we must again turn to a non-relativistic reduction but now in the case of a classical vector potential of the form $A^\mu_{cl}(\vec{x}) = (\phi(\vec{x}), \vec{0})$. The Fourier transform of this A^μ_{cl} is again what appears in $i\mathcal{M}$ and so we have, referring to Eq. (386):

$$i\mathcal{M} = -ie\overline{u}(p')\Gamma^0 u(p)\widetilde{\phi}(\vec{q})\,. \tag{400}$$

For definition of the charge, we want to consider a potential that is essentially constant as a function of $\vec{x}$ over some large region in which the charge is being probed. In this case, $\widetilde{\phi}(\vec{q})$ will be concentrated at $\vec{q} = 0$. Thus, it is appropriate to take $\vec{q} \to 0$ in defining the charge, and the F_2 term then drops out and we evaluate F_1 at $\vec{q} = 0$. Then, we only need to compute

$$\overline{u}(p')\gamma^0 u(p) = u^\dagger(p')u(p) \to 2m\xi'^\dagger\xi \tag{401}$$

so that we end up with

$$i\mathcal{M} = -ieF_1(0)\widetilde{\phi}(\vec{q})2m\xi'^\dagger\xi \tag{402}$$

which is the Born approximation for scattering from a potential of the form

$$V(\vec{x}) = eF_1(0)\phi(\vec{x})\,. \tag{403}$$

This means that $F_1(0)$ is the electric charge of the electron in units of e, which by definition means that $F_1(0) = 1$ is required after carrying out the renormalization program. Since $F_1(0) = 1$ at the 0-loop level (tree-level), radiative corrections to $F_1(q^2)$ should vanish as $q^2 \to 0$.

To compute $F_2(0)$, we first identify that part of $\Lambda_\mu^{(2)}$ that gives rise to the $i\sigma_{\mu\nu}q^\nu$ structure that defines F_2. Referring to Eq. (315), we find

$$\overline{u}(p')\Lambda_\mu^{(2)}u(p) = \overline{u}(p')\left\{\frac{e^2}{16\pi^2}\int_0^1 d\alpha\int_0^{1-\alpha} d\beta\frac{\gamma_\nu(\not p'(1-\beta)-\alpha\not p+m)\gamma_\mu(\not p(1-\alpha)-\beta\not p'+m)\gamma^\nu}{[-m^2(\alpha+\beta)+\alpha(1-\alpha)p^2+\beta(1-\beta)p'^2-\alpha\beta 2p'\cdot p]}\right\}u(p) \tag{404}$$

To proceed, we need to do some work on the numerator. Remembering that $\gamma_\nu \not a \not b\gamma^\nu = 4a\cdot b$, $\gamma_\nu \not a \not b \not c\gamma^\nu = -2\not c \not b \not a$, $\gamma_\nu \not a\gamma^\nu = -2\not a$, we have

$$\begin{aligned}
&\gamma_\nu(\not p'(1-\beta)-\alpha\not p+m)\gamma_\mu(\not p(1-\alpha)-\beta\not p'+m)\gamma^\nu \\
=\ & -2(\not p(1-\alpha)-\beta\not p')\gamma_\mu(\not p'(1-\beta)-\alpha\not p)+4m(p_\mu(1-\alpha)-\beta p'_\mu)+4m(p'_\mu(1-\beta)-\alpha p_\mu)-2\gamma_\mu m^2 \\
\to\ & -2(\not p(1-\alpha)-\beta m)\gamma_\mu(\not p'(1-\beta)-\alpha m)+4m(p_\mu(1-\alpha)-\beta p'_\mu)+4m(p'_\mu(1-\beta)-\alpha p_\mu)-2\gamma_\mu m^2
\end{aligned} \tag{405}$$

where the $\to$ indicates that we used the Dirac equation for the spinors.

We now need a couple of intermediate results.

$$\begin{aligned} \not{p}\gamma_\mu \not{p}' &= 2p_\mu \not{p}' - \gamma_\mu \not{p} \not{p}' \\ &= 2p_\mu \not{p}' - \gamma_\mu 2p \cdot p' + \gamma_\mu \not{p}' \not{p} \\ &= 2p_\mu \not{p}' - \gamma_\mu 2p \cdot p' + 2p'_\mu \not{p} - \not{p}'\gamma_\mu \not{p} \\ &\rightarrow 2p_\mu m - \gamma_\mu 2p \cdot p' + 2p'_\mu m - \gamma_\mu m^2 \\ &= 2(p_\mu + p'_\mu)m - 3m^2\gamma_\mu \end{aligned} \tag{406}$$

where in the last step we used the kinematics $p^2 = m^2$, $p'^2 = m^2$ and $(p - p')^2 = 2m^2 - 2p \cdot p' = q^2 = 0$ which implies $2p \cdot p' = 2m^2$. We also have

$$\not{p}\gamma_\mu = 2p_\mu - \gamma_\mu \not{p} \rightarrow 2p_\mu - \gamma_\mu m\,, \tag{407}$$

$$\gamma_\mu \not{p}' = 2p'_\mu - \not{p}'\gamma_\mu \rightarrow 2p'_\mu - \gamma_\mu m\,. \tag{408}$$

Altogether, we get for the numerator

$$N = -2\Big[(1-\alpha)(1-\beta)(2(p+p')_\mu m - 3m^2\gamma_\mu) - \beta(1-\beta)m(2p'_\mu - m\gamma_\mu)$$

$$-\alpha(1-\alpha)m(2p_\mu - m\gamma_\mu) + \alpha\beta m^2\Big] + 4m\left[p'_\mu(1-2\beta) + p_\mu(1-2\alpha)\right] - 2\gamma_\mu m^2$$

$$\begin{aligned} = \quad & -2\gamma_\mu m^2[-3(1-\alpha)(1-\beta) + \beta(1-\beta) + \alpha(1-\alpha) + \alpha\beta + 1] \\ & +4p_\mu m[\beta - \alpha\beta - \alpha^2] + 4p'_\mu m[\alpha - \beta\alpha - \beta^2] \end{aligned} \tag{409}$$

Meanwhile, the denominator takes the form (using the same kinematics as already discussed)

$$D = -m^2(\alpha+\beta)^2 . \tag{410}$$

Because of the symmetry of the denominator, we can rewrite N in the form

$$N = -2\gamma_\mu m^2[4(\beta+\alpha) - (\alpha+\beta)^2 - 2] + 2(p_\mu + p'_\mu)m[(\alpha+\beta) - (\alpha+\beta)^2] \tag{411}$$

so that we now have

$$\begin{aligned} & \overline{u}(p')\Lambda^{(2)}_\mu u(p) \\ = \quad & \overline{u}(p')\frac{-e^2}{16\pi^2 m^2}\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{1}{(\alpha+\beta)^2} \\ & \left(-2\gamma_\mu m^2[4(\beta+\alpha) - (\alpha+\beta)^2 - 2] + 2(p_\mu + p'_\mu)m[(\alpha+\beta) - (\alpha+\beta)^2]\right) u(p) \end{aligned}$$

$$= \overline{u}(p')\frac{-e^2}{16\pi^2 m^2}\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{1}{(\alpha+\beta)^2}$$
$$\left(-2\gamma_\mu m^2[4(\beta+\alpha)-(\alpha+\beta)^2-2]+2[2m\gamma_\mu - i\sigma_{\mu\nu}q^\nu]m[(\alpha+\beta)-(\alpha+\beta)^2]\right)u(p)$$
$$= \overline{u}(p')\frac{-e^2}{16\pi^2 m^2}\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{1}{(\alpha+\beta)^2}$$
$$\left(-2\gamma_\mu m^2[2(\beta+\alpha)+(\alpha+\beta)^2-2]-2i\sigma_{\mu\nu}q^\nu m[(\alpha+\beta)-(\alpha+\beta)^2]\right)u(p)\,. \tag{412}$$

We actually don't care about the γ_μ part. For the anomalous magnetic moment we only want the coefficient of $+\frac{i\sigma_{\mu\nu}}{2m}$ which contains the two integrals

$$\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{1}{(\alpha+\beta)} = \int_0^1 d\alpha \ln(\alpha+\beta)_{\beta=0}^{\beta=1} = -\int_0^1 d\alpha \ln\alpha = -[\alpha\ln\alpha - \alpha]_0^1 = 1 \tag{413}$$

and

$$\int_0^1 d\alpha \int_0^{1-\alpha} d\beta = \frac{1}{2} \tag{414}$$

yielding

$$\frac{i\sigma_{\mu\nu}q^\nu}{2m}\left[\frac{e^2}{8\pi^2}\right] = \frac{i\sigma_{\mu\nu}q^\nu}{2m}\left[\frac{\alpha}{2\pi}\right]\,. \tag{415}$$

The above is the well-known result from Schwinger in 1948 for the anomalous magnetic moment addition to bare contribution yielding a total of

$$\frac{i\sigma_{\mu\nu}q^\nu}{2m}\left[1+\frac{\alpha}{2\pi}\right]. \tag{416}$$

where the relation to the usual g_e is

$$\frac{g_e}{2}=1+\frac{\alpha}{2\pi}. \tag{417}$$

Please note that I got the correct sign for my sign of the denominator and that Ryder slipped in an extra sign. At the present, $\frac{g_e}{2}$ has been calculated to order α^3, yielding impressive agreement between theory and experiment.

$$\begin{aligned} a_{th} &= \frac{1}{2}(g_e-2) \\ &= (1159652.4\pm 0.4)\times 10^{-9} \\ a_{exp} &= (1159652.4\pm 0.2)\times 10^{-9}. \end{aligned} \tag{418}$$

Actually, experiment has gotten a bit better recently and it is time to do the

$\mathcal{O}(\alpha^4)$ calculation. This is the kind of agreement that makes us think that this whole business of QED and renormalization really does make sense.

How to use the Ward identity in the proof of renormalizability

I will gives just a simple illustration of why the WI is so useful: Overlapping Divergences.

Consider the diagram for $-i\Sigma$ that corresponds to two overlapping virtual photon loops. Labeling the photon vertices as ρ, μ, ρ, μ (after already using the $g_{\ldots}$ of the photon propagators) and using k_1 and k_2 for the $\mu \to \mu$ photon and the $\rho \to \rho$ photon, respectively, the expression for the diagram takes the form

$$\int d^4k_1 d^4k_2 \frac{1}{k_1^2}\frac{1}{k_2^2}\gamma^\rho \frac{1}{\not p - \not k_2 - m}\gamma^\mu \frac{1}{\not p - \not k_1 - \not k_2 - m}\gamma_\rho \frac{1}{\not p - \not k_1 - m}\gamma_\mu \,, \tag{419}$$

which is superficially $D = 1$ divergent. If k_1 is held fixed, then the d^4k_2 integral has $D = 0$. In other words the overlapping divergences from the two integrations cannot be separated out. The WI allows one to circumvent this difficulty. The idea is to remember that the WI says that $\frac{\partial \Sigma}{\partial p^\alpha} = -\Lambda_\alpha$ and to begin by showing that in Λ_α the divergences can be separated. In fact, this latter separation is always possible. There are 3 associated Λ_α

graphs corresponding to the 3 different locations in the Σ graph at which a photon can be attached on an internal electron propagator. Let us consider, for example, the diagram in which the photon (index α) is attached in the left-most location. For our purposes, the photon can have $q = 0$. The expression for this diagram takes the form

$$\int d^4k_1 d^4k_2 \frac{1}{k_1^2}\gamma^\rho \frac{1}{\not p - \not k_2 - m}\gamma^\mu \frac{1}{\not p - \not k_1 - \not k_2 - m}\gamma_\rho \frac{1}{\not p - \not k_1}\gamma_\alpha \frac{1}{\not p - \not k_1}\gamma_\mu \frac{1}{k_2^2}. \tag{420}$$

Now, for fixed k_2, we have $\int \frac{d^4k_1}{k_1^5}$ which is convergent and only $\int d^4k_2$ must be dealt with, but this can be done using the usual procedures (dimensional regularization). The story is the same for all the diagrams. But, now we can figure out how to write Σ by using the WI differential equation.

At this point, I could discuss the 2-loop calculation for the photon propagator in Itzykson and Zuber (Euclidean conventions). (This begins on p. 7 of my hand-written notes.) However, I don't believe we have time for this.

Renormalization of QCD

Because the gluon-quark-quark and 3-gluon vertex are both given by g times appropriate structure, we could compute the 1-loop corrections to either in order to figure out how g behaves as a function of the scale μ. Hopefully, they would continue to be tied together at 1-loop. This is, indeed, the case. For us, it is simpler to use the gluon-quark-quark vertex to define g since then we can relate the required calculations in part to those already done for QED. We will do our calculations in Feynman gauge, which means that at some points we will have to consider ghosts. Relevant Feynman rules are on p. 220 of Path Integral Notes pdf file.

$-i\Sigma$

As in QED, however, we first begin with $-i\Sigma^{AB}$, where A, B are the initial and final quark color indices, which is very simply related to the QED calculation already performed. We have:

$$- i\Sigma^{AB} = \int \frac{d^n k}{(2\pi)^n} (ig\mu^\epsilon \gamma_\mu) \frac{i}{\not{p} - \not{k} - m} (ig\mu^\epsilon \gamma_\nu) \frac{-ig^{\mu\nu}}{k^2} (T^c)_{AD} (T^c)_{DB} \,. \tag{421}$$

We already know the result of the color algebra is to give $C_F\delta^{AB}$. Thus, we very quickly obtain

$$-i\Sigma^{AB} = C_F\delta^{AB} \times -i\Sigma(QED) = -i\frac{g^2}{16\pi^2\epsilon}(-\not{p} + 4m)C_F\delta^{AB}\,. \quad (422)$$

Using $C_F = 4/3$ for $SU(3)$, and using the $\not{p}$ coefficient to determine Z_2, we obtain

$$Z_2 = 1 - \frac{g^2}{16\pi^2\epsilon}C_F = 1 - \frac{g^2}{12\pi^2\epsilon}\,. \quad (423)$$

$i\Pi$

The one loop diagrams are: gluon-loop insertion; ghost-loop insertion; fermion-loop insertion; gluon loop quartic vertex "tadpole"; gluon-loop true tadpole; ghost-loop true tadpole; the fermion-loop true tadpole. The loop integration momentum will typically be called p and the external gluon momentum will be called k.

The fermion-loop true tadpole is 0 for exactly the same reason as in QED — $\int \frac{d^np}{(2\pi)^n}\frac{\mathrm{Tr}[\gamma_\alpha(\not{p}+m)]}{p^2-m^2} = 0$ due to $\mathrm{Tr}[\gamma_\alpha] = 0$ and oddness of $\mathrm{Tr}[\gamma_\alpha\not{p}] = 4p_\alpha$ under $p \to -p$, whereas the denominator is even. Here α was the Lorentz index of the virtual $q = 0$ gluon running from the propagating gluon and attaching to the fermion loop.

The gluon-loop true tadpole and ghost-loop true tadpole are both 0 for the same basic reason: they both are proportional to $\int \frac{d^n p}{(2\pi)^n}\frac{1}{p^2} \times p^\alpha = 0$.

The quartic vertex "tadpole" is 0 for a more subtle reason. The gluon loop integral does not contain any numerator momenta. Thus, the integral in question is

$$\int \frac{d^n p}{(2\pi)^n}\frac{1}{p^2}\,. \tag{424}$$

To evaluate this, we temporarily introduce an m^2 and then later take $m^2 \to 0$. We have, using our master formula,

$$\begin{aligned}\int \frac{d^n p}{(2\pi)^n}\frac{1}{p^2 - m^2} &= (-1)^{n/2}\frac{i}{(4\pi)^{n/2}}\frac{\Gamma(1-\frac{n}{2})}{\Gamma(1)}[-m^2]^{n/2-1} \\ &\stackrel{m^2\to 0}{\longrightarrow} 0\,,\end{aligned} \tag{425}$$

provided that $n/2 - 1 > 0$, which is of course the case for $n \to 4$. Note that for $n \to 2$, we cannot use this argument and there is in fact a pole at $n \to 2$ reflecting the basic quadratic divergence of this integral.

To make this more explicit and to illustrate some important lessons we evaluate the basic integral using a different set of tricks. We write

$$\begin{aligned}
\int \frac{d^n p}{(2\pi)^n}\frac{1}{p^2} &= \int \frac{d^n p}{(2\pi)^n}\frac{1}{p^2}\frac{(p+k)^2}{(p+k)^2} \\
&= \int d\alpha \int \frac{d^n p}{(2\pi)^n}\frac{(p+k)^2}{[\alpha(p+k)^2+(1-\alpha)p^2]^2} \\
&= \int d\alpha \int \frac{d^n p'}{(2\pi)^n}\frac{(p'+(1-\alpha)k)^2}{[p'^2+\alpha(1-\alpha)k^2]^2} \quad \text{using shift to } p'=p+\alpha k \\
&= \int d\alpha \int \frac{d^n p'}{(2\pi)^n}\left[\frac{1}{[p'^2+\alpha(1-\alpha)k^2]}+\frac{(1-\alpha)(1-2\alpha)k^2}{[p'^2+\alpha(1-\alpha)k^2]^2}\right] \quad \text{dropping terms odd in } p' \\
&= \int d\alpha (-1)^{n/2}\frac{i}{(4\pi)^{n/2}}\left[\frac{\Gamma(1-\frac{n}{2})}{\Gamma(1)}\frac{1}{[\alpha(1-\alpha)k^2]^{1-\frac{n}{2}}}+\frac{\Gamma(2-\frac{n}{2})}{\Gamma(2)}\frac{(1-\alpha)(1-2\alpha)k^2}{[\alpha(1-\alpha)k^2]^{2-\frac{n}{2}}}\right] \\
&= \int d\alpha (-1)^{n/2}\frac{i}{(4\pi)^{n/2}}\frac{1}{[\alpha(1-\alpha)k^2]^{2-\frac{n}{2}}}\left[\Gamma(1-\frac{n}{2})k^2\frac{n}{2}\alpha(1-\alpha)+\Gamma(2-\frac{n}{2})k^2(1-\alpha)^2\right]
\end{aligned} \tag{426}$$

Now, as $n \longrightarrow 4$, we can still check that we get 0 by virtue of cancellation of the two terms. We must use (as $n \longrightarrow 4$)

$$\begin{aligned}
\Gamma(1-\tfrac{n}{2}) &= \frac{\Gamma(2-\frac{n}{2})}{1-\frac{n}{2}} \sim -\Gamma(2-\tfrac{n}{2}\,,) \\
\int d\alpha\,\alpha(1-\alpha) &= \tfrac{1}{6}\,, \quad \int d\alpha (1-\alpha)^2 = \tfrac{1}{3} \quad .
\end{aligned} \tag{427}$$

But, we want to keep the form given so that we can see how this term combines with the other 1-loop Π contributions when $n \neq 4$. In particular, we want to see that we get a gauge-invariant structure *even when* $n \neq 4$, thereby demonstrating that dimensional regularization preserves gauge invariance.

Of course, we have not written down this "tadpole" contribution with all the correct factors and couplings and color structure. To see what we get when we do that, we start with our basic Feynman expression, including color factors, in the form (using gluon color labels ordered clockwise as a, d, c, b going with μ, σ, ρ, ν — also note contraction symmetry factor of $1/2$ and use $g_\rho^\rho = n$):

$$\begin{aligned}
i\Pi^{ab}_{\mu\nu} \ni\ & \frac{1}{2}\int \frac{d^n p}{(2\pi)^n}\frac{-ig_{\rho\sigma}}{p^2}\delta^{cd}(ig^2) \\
& \left[ic^{dce}ic^{eba}(g^{\sigma\nu}g^{\mu\rho} - g^{\mu\sigma}g^{\nu\rho}) + ic^{dbe}ic^{eca}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + ic^{dae}ic^{ebc}(g^{\sigma\nu}g^{\rho\mu} - g^{\sigma\rho}g^{\mu\nu})\right] \\
=\ & g^2\int \frac{d^n p}{(2\pi)^n}\frac{1}{p^2}\left[0 + (-C_A\delta^{ab})[(n-1)g_{\mu\nu}] + (C_A\delta^{ab})[-(n-1)g_{\mu\nu}]\right] \\
=\ & -g^2 C_A\delta^{ab}(-1)^{n/2}\frac{i}{(4\pi)^{n/2}}\int d\alpha \frac{(n-1)g_{\mu\nu}}{[\alpha(1-\alpha)k^2]^{2-\frac{n}{2}}}\left[\Gamma(1-\frac{n}{2})k^2\frac{n}{2}\alpha(1-\alpha) + \Gamma(2-\frac{n}{2})k^2(1-\alpha)^2\right]
\end{aligned} \tag{428}$$

where the color diagram created by $\delta^{cd}ic^{dae}ic^{ebc} \Rightarrow +C_A\delta^{ab}$ (no crossed gluons) whereas that created by $\delta^{cd}ic^{dbe}ic^{eca}$ has a crossed gluon configuration that must be "unwrapped" (leading to a minus sign because of $c^{\cdots}$ antisymmetry) to obtain a diagram that leads to $+C_A\delta^{ab}$.

So, now we must work on the other diagrams.

The fermion-loop diagram works just like QED except for the extra color factor of $\frac{1}{2}$ from our representation matrix product trace and a factor of n_f which is the number of different "flavors" of quarks that should be put into the fermion loop. Referring back to Eq. (299), we have (for the singular piece):

$$i\Pi^{ab}_{\mu\nu} \ni \frac{ig^2}{12\pi^2}\delta^{ab}(k_\mu k_\nu - k^2 g_{\mu\nu})\Gamma(2-\frac{n}{2})(\frac{1}{2}n_f)\,, \tag{429}$$

where as $\epsilon \to 0$, $\Gamma(2-\frac{n}{2}) = \Gamma(\epsilon) \to \frac{1}{\epsilon}$.

Next comes the gluon-loop diagram. Here, the color calculation just gives us $C_A\delta^{ab}$. Below, we discuss the Feynman-momentum space calculation. Starting with a gluon vertex μ,ρ,σ (moving clockwise) on the left and ρ',ν,σ' on the right, and using p flowing into the left-hand vertex along the (bottom) σ,σ' gluon line and $p+k$ flowing out of the left-hand vertex along the (top) ρ,ρ' line, we have

$$i\Pi^{ab}_{\mu\nu} \quad = \quad C_A\delta^{ab}\frac{(-ig\mu^\epsilon)^2}{2}\int\frac{d^np}{(2\pi)^n}\frac{N_{\mu\nu}}{(p+k)^2k^2}$$

$$= \quad C_A \delta^{ab} \frac{(-ig\mu^{\epsilon})^2}{2} \int d\alpha \int \frac{d^n p}{(2\pi)^n} \frac{N_{\mu\nu}}{[(p+\alpha k)^2 + \alpha(1-\alpha)k^2]^2} \tag{430}$$

where

$$\begin{aligned}
N_{\mu\nu} \quad &= \quad [g_{\mu\sigma}(-k+p)_\rho + g_{\mu\rho}(p+2k)_\sigma + g_{\rho\sigma}(-2p-k)_\mu] \times \\
&\quad\quad [g_{\sigma'\nu}(p-k)_{\rho'} + g_{\rho'\sigma'}(-2p-k)_\nu + g_{\rho'\nu}(p+2k)_{\sigma'}](-ig^{\rho\rho'})(-ig^{\sigma\sigma'}) \\
&= \quad -g_{\mu\nu}[5k^2 + 2p^2 + 2k\cdot p] - (n-6)k_\mu k_\nu - (4n-6)p_\mu p_\nu - (2n-3)(p_\mu k_\nu + p_\nu k_\mu) \\
&\underset{\longrightarrow}{p=p'-\alpha k} \quad -g_{\mu\nu}[k^2(5-2\alpha(1-\alpha)) + 6(p')^2\frac{n-1}{n}] - k_\mu k_\nu[(6-4n)\alpha(1-\alpha)+n-6] \\
&\underset{\longrightarrow}{\int d^n p'} \quad (-1)^{n/2}\frac{i}{(4\pi)^{n/2}}\frac{1}{[\alpha(1-\alpha)k^2]^\epsilon}\left\{-g_{\mu\nu}\left[k^2[5-2\alpha(1-\alpha)]\Gamma(2-\frac{n}{2})\right.\right. \\
&\quad\quad \left.\left. +3(n-1)k^2\alpha(1-\alpha)\Gamma(1-\frac{n}{2})\right] - k_\mu k_\nu\left([6-4n]\alpha(1-\alpha)+n-6\right)\Gamma(2-\frac{n}{2})\right\} \\
&\underset{\longrightarrow}{n\to 4} \quad \frac{i}{(4\pi)^2}\Gamma(\epsilon)\left\{-g_{\mu\nu}\left(k^2[5-11\alpha(1-\alpha)]\right) - k_\mu k_\nu\left(-10\alpha(1-\alpha)-2\right)\right\} \\
&\underset{\longrightarrow}{\int d\alpha} \quad \frac{i}{(4\pi)^2}\Gamma(\epsilon)\left\{-\frac{19}{6}g_{\mu\nu}k^2 + \frac{11}{3}k_\mu k_\nu\right\},
\end{aligned} \tag{431}$$

where we isolated only the singular part. This leads to

$$i\Pi^{ab}_{\mu\nu} = C_A\delta^{ab}\left(\frac{-g^2}{2}\right)\frac{i}{(4\pi)^2}\Gamma(\epsilon)\left\{-\frac{19}{6}g_{\mu\nu}k^2+\frac{11}{3}k_\mu k_\nu\right\}, \quad (432)$$

which can be compared to Ryder's result. However, we really want to back track to the expression obtained after $\int d^n p'$. At that stage our result is

$$i\Pi^{ab}_{\mu\nu} = C_A\delta^{ab}\left(\frac{-g^2}{2}\right)\int d\alpha(-1)^{n/2}\frac{i}{(4\pi)^{n/2}}\frac{1}{[\alpha(1-\alpha)k^2]^\epsilon}\left\{-g_{\mu\nu}\left[k^2[5-2\alpha(1-\alpha)]\Gamma(2-\frac{n}{2})\right.\right.$$
$$\left.\left.+3(n-1)k^2\alpha(1-\alpha)\Gamma(1-\frac{n}{2})\right]-k_\mu k_\nu\left([6-4n]\alpha(1-\alpha)+n-6\right)\Gamma(2-\frac{n}{2})\right\} \quad (433)$$

So, finally, we have to compute the ghost-loop diagram. You will notice that if we sum the "tadpole" and the gluon-loop diagram, we do not have the required gauge invariant structure yet. Also, both diagrams have a $\Gamma(1-\frac{n}{2})$ piece (reflecting a quadratic divergence) and these do not cancel, suggesting that we could have a hierarchy problem. The ghost-loop diagram will take care of both problems.

The expression we must evaluate is (remember $-$ sign for ghost loop)

$$
\begin{aligned}
i\Pi^{ab}_{\mu\nu} &= (-1)C_A\delta^{ab}\int\frac{d^np}{(2\pi)^n}\frac{i}{p^2}\frac{i}{(p+k)^2}(-ig)^2(p+k)_\mu p_\nu \\
&= -g^2C_A\delta^{ab}\int d\alpha\int\frac{d^np}{(2\pi)^n}\frac{(p+k)_\mu p_\nu}{[(p+\alpha k)^2+\alpha(1-\alpha)k^2]^2} \\
&= -g^2C_A\delta^{ab}\int d\alpha\int\frac{d^np'}{(2\pi)^n}\frac{(p')_\nu(p')_\mu-\alpha(1-\alpha)k_\mu k_\nu}{[(p')^2+\alpha(1-\alpha)k^2]^2} \quad \text{dropping terms odd in } p' \\
&= -g^2C_A\delta^{ab}\int d\alpha(-1)^{n/2}\frac{i}{(4\pi)^{n/2}}\frac{1}{[\alpha(1-\alpha)k^2]^\epsilon}\left[\frac{1}{2}g_{\mu\nu}k^2\alpha(1-\alpha)\Gamma(1-\frac{n}{2})-\alpha(1-\alpha)k_\mu k_\nu\Gamma(2-\frac{n}{2})\right]
\end{aligned}
\tag{434}
$$

We are now in a position to collect our results together. Let us first examine the $\Gamma(1-\frac{n}{2})$ terms. These are

$$
\begin{aligned}
i\Pi^{ab}_{\mu\nu} &\ni g_{\mu\nu}(-1)^{n/2}\frac{ig^2}{(4\pi)^{n/2}}C_A\delta^{ab}\Gamma(1-\frac{n}{2})\int d\alpha\frac{k^2\alpha(1-\alpha)}{[k^2\alpha(1-\alpha)]^\epsilon}\left[-(n-1)\frac{n}{2}+\frac{3(n-1)}{2}-\frac{1}{2}\right] \\
&= g_{\mu\nu}(-1)^{n/2}\frac{ig^2}{(4\pi)^{n/2}}C_A\delta^{ab}\Gamma(1-\frac{n}{2})\int d\alpha\frac{k^2\alpha(1-\alpha)}{[k^2\alpha(1-\alpha)]^\epsilon}\left[(1-\frac{n}{2})(n-2)\right] \\
&= g_{\mu\nu}(-1)^{n/2}\frac{ig^2}{(4\pi)^{n/2}}C_A\delta^{ab}\Gamma(2-\frac{n}{2})\int d\alpha\frac{k^2\alpha(1-\alpha)}{[k^2\alpha(1-\alpha)]^\epsilon}(n-2)\,.
\end{aligned}
\tag{435}
$$

Note that all the singularities at $1-\frac{n}{2}=0$ from $\Gamma(1-\frac{n}{2})$ have been canceled by the explicit factor of $1-\frac{n}{2}$ that emerged after summing all the gauge-related diagrams. We must now combine the residual above with all the other $g_{\mu\nu}$ terms.

We find

$$
\begin{aligned}
i\Pi^{ab}_{\mu\nu} \quad &= \quad (-1)^{n/2}\frac{ig^2}{(4\pi)^{n/2}}C_A\delta^{ab}\Gamma(2-\frac{n}{2})\int\frac{d\alpha}{[k^2\alpha(1-\alpha)]^\epsilon} \\
&\left\{ g_{\mu\nu}k^2\left[-(n-1)(1-\alpha)^2+\frac{1}{2}[5-2\alpha(1-\alpha)]+(n-2)\alpha(1-\alpha)\right]\right. \\
&\left. +k_\mu k_\nu\left[0+\frac{1}{2}\left[(6-4n)\alpha(1-\alpha)+n-6\right]+\alpha(1-\alpha)\right]\right\} \qquad (436)
\end{aligned}
$$

In the $g_{\mu\nu}$ coefficient expression, the 1st term comes from the $\Gamma(2-\frac{n}{2})$ "tadpole" terms, the 2nd term from the gluon loop stuff proportional to $\Gamma(2-\frac{n}{2})$ and the last term is the residual after summing all the $\Gamma(1-\frac{n}{2})$ stuff so that it reduces to $\Gamma(2-\frac{n}{2})$ as described just above. In the $k_\mu k_\nu$ coefficient expression, the 1st 0 entry is for the "tadpole" graph, the 2nd entry is for the gluon-loop graph, and the 3rd term is the $\Gamma(2-\frac{n}{2})$ part of the ghost-loop graph.

The above expression can be simplified by making the $g_{\mu\nu}$ coefficient explicitly symmetric in $\alpha \to 1-\alpha$ using the symmetry of the denominator. This means we want to write

$$(1-\alpha)^2 = \frac{1}{2}\left[(1-\alpha)^2 + \alpha^2\right] = \frac{1}{2} - \alpha(1-\alpha)\,. \tag{437}$$

Inserting this result and doing a bit more algebra, we obtain:

$$i\Pi^{ab}_{\mu\nu} \quad = \quad (-1)^{n/2}\frac{ig^2}{(4\pi)^{n/2}}C_A\delta^{ab}\Gamma(2-\frac{n}{2})\int\frac{d\alpha}{[k^2\alpha(1-\alpha)]^\epsilon}\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\left[(1-\frac{n}{2})(1-2\alpha)^2 + 2\right] \tag{438}$$

Note that after summing all the contributions to $i\Pi^{ab}_{\mu\nu}$, we obtain an expression that has manifest gauge invariance built in, *i.e.* $k^\mu i\Pi^{ab}_{\mu\nu} = 0$. We new this had to happen, but as you see it is very essential to combine all contributions in order to arrive at a GI invariant form.

It is now easy to isolate the $\frac{1}{\epsilon}$ singularity of the above expression. We find:

$$i\Pi^{ab}_{\mu\nu} \to \frac{1}{\epsilon}\frac{ig^2}{(4\pi)^2}C_A\delta^{ab}\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\left(\frac{5}{3}\right)\,. \tag{439}$$

We shall see how this fits into things later on, but you should note how this compares to our QED result

$$i\Pi_{\mu\nu} \to \frac{1}{\epsilon}\frac{ie^2}{(4\pi)^2}\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\left(-\frac{4}{3}\right). \qquad (440)$$

Note the opposite sign. This is the signal that QCD will be (or at least can be, assuming not too many flavors of fermions) asymptotically free whereas QED was the opposite.

We now bring in the fermion-loop contribution to the $\frac{1}{\epsilon}$ singularity [see Eq. (429)] and find

$$i\Pi^{ab}_{\mu\nu} \to \frac{1}{\epsilon}\frac{ig^2}{(4\pi)^2}\delta^{ab}\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\left(\frac{5}{3}C_A - \frac{4}{3}(\frac{1}{2}n_f)\right), \qquad (441)$$

where sometimes $\frac{1}{2}n_f$ is written as $C(r)n_f$ where $C(r)$ expresses the normalization of the fermion representation matrices, which we know have traces normalized to $\frac{1}{2}$ in the fundamental representation that we have been assuming.

Finally, we can immediately read off from the above what the the gluon field renormalization factor is. Recall the procedure. We introduce a counter-term

$\mathcal{L}$ component of the form

$$\Delta\mathcal{L} \ni -\frac{C}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\,\nu} - \partial^\nu A^{a\,\mu}) \tag{442}$$

which leads to Feynman rule

$$-iC(k^2 g_{\mu\nu} - k_\mu k_\nu)\delta^{ab} \tag{443}$$

and we must choose C so that the sum of this and the $\frac{1}{\epsilon}$ component of $i\Pi^{ab}_{\mu\nu}$ is finite. This clearly leads to

$$C = \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\left(\frac{5}{3}C_A - \frac{4}{3}(\frac{1}{2}n_f)\right) . \tag{444}$$

The net (quadratic in A) part of the Lagrangian form is then

$$\mathcal{L}+\Delta\mathcal{L} \ni -\frac{1+C}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\,\nu} - \partial^\nu A^{a\,\mu}) \equiv -\frac{Z_3}{4}(\partial_\mu A^{0\,a}_\nu - \partial_\nu A^{0\,a}_\mu)(\partial^\mu A^{a\,0\,\nu} - \partial^\nu A^{a\,0\,\mu}) \tag{445}$$

leading to

$$Z_3 = 1 + \frac{g^2}{(4\pi)^2\epsilon}\left(\frac{5}{3}C_A - \frac{4}{3}(\frac{1}{2}n_f)\right) . \tag{446}$$

$i\Lambda$

We must now work on the gluon-quark-quark vertex. There are two diagrams. In one, the gluon attaches to the quark line of the gluon loop insertion on a quark line. In the second, the gluon attaches to the gluon line. This result for the first can easily be read off of our earlier QED result. Everything is just as in QED except that we have the group calculation that we did some time ago, which found that the appropriate group factor was $-\frac{1}{2n}$ which can also be written as $C_F - \frac{1}{2}C_A$. We must also remember that the basic vertex in our notation is

$$+ig\mu^\epsilon\gamma_\mu \times (L^a_{AB}) \tag{447}$$

where L^a is the representation matrix and we have supplied the appropriate dimensional factor for n dimensions. Thus, the appropriate notation for the one-loop correction is to write

$$+ig\mu^\epsilon\Lambda_\mu \times (L^a_{AB}). \tag{448}$$

After introducing the above color factor, the divergent component coming from the fermion attachment graph is

$$\Lambda_\mu = \frac{g^2}{16\pi^2\epsilon}\gamma_\mu(C_F - \frac{1}{2}C_A)\,. \tag{449}$$

Now, let me show you a quick a dirty approach to rederiving this result. For the momentum space component of the calculation we have (neglecting the electron mass for simplicity):

$$\begin{aligned}
+ig\mu^\epsilon\Lambda_\mu &= (ig\mu^\epsilon)(ig\mu^\epsilon)^2\int\frac{d^nk}{(2\pi)^n}\gamma^\nu\frac{i}{\not p' + \not k}\gamma_\mu\frac{i}{\not p + \not k}\gamma^\lambda\left(\frac{-ig_{\nu\lambda}}{k^2}\right)\\
&= (ig\mu^\epsilon)(ig\mu^\epsilon)^2 i\int\frac{d^nk}{(2\pi)^n}\frac{\gamma^\nu(\not p' + \not k)\gamma_\mu(\not p + \not k)\gamma_\nu}{(p'+k)^2(p+k)^2k^2}\\
&\to (ig\mu^\epsilon)i(ig)^2\int\frac{d^nk}{(2\pi)^n}\frac{\gamma^\nu\not k\gamma_\mu\not k\gamma_\nu}{(k^2)^3} \quad \text{keeping only the most divergent stuff}\\
&= (ig\mu^\epsilon)(-ig^2)\int\frac{d^nk}{(2\pi)^n}(2-n)\frac{\not k\gamma_\mu\not k}{(k^2)^3}\\
&= (ig\mu^\epsilon)(-ig^2)\int\frac{d^nk}{(2\pi)^n}(2-n)\frac{\gamma^\rho\gamma_\mu\gamma^\sigma k_\rho k_\sigma}{(k^2)^3}\\
&= (ig\mu^\epsilon)(-ig^2)\int\frac{d^nk}{(2\pi)^n}(2-n)\frac{\gamma^\rho\gamma_\mu\gamma^\sigma\frac{g_{\rho\sigma}}{n}k^2}{(k^2)^3}\\
&= (ig\mu^\epsilon)(-ig^2)\int\frac{d^nk}{(2\pi)^n}\frac{(2-n)^2}{n}\gamma_\mu\frac{1}{(k^2)^2}
\end{aligned}$$

$$
\begin{aligned}
&= (ig\mu^\epsilon)(-ig^2)(-1)^{n/2}\frac{i}{(4\pi)^{n/2}}\frac{(2-n)^2}{n}\Gamma(2-\frac{n}{2})\gamma_\mu \\
&\to (ig\mu^\epsilon)\frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\gamma_\mu\,,
\end{aligned} \tag{450}
$$

which reproduces the expected result once the color factor of $C_F - \frac{1}{2}C_A$ is supplied. The reason we could not use this technique for $i\Pi$ is that the contributing diagrams had quadratic $\Gamma(1-\frac{n}{2})$ divergences that had to be combined to get the residual $\Gamma(2-\frac{n}{2})$ logarithmic divergence. In the present case, the superficial divergence of each diagram is from the beginning only logarithmic, $\Gamma(2-\frac{n}{2})$. Thus, we may attack the final diagram in which the gluon attaches to the gluon in the triangle graph in the same simplified manner. The diagram notation is that the fermion flows into the diagram with momentum p, encounters vertex γ_ν after which it changes to momentum k, then encounters γ_λ and turns into p'. The three-gluon vertex has momentum $p-k$ entering from left with associated index σ which exits with momentum $p'-k$ with associated index ρ. Meanwhile, the external gluon entering from the top has momentum $p'-p$ and index μ. The resulting non-color part of the calculation is:

$$
\begin{aligned}
ig\mu^\epsilon\Lambda_\mu &= \int\frac{d^nk}{(2\pi)^n}(ig\mu^\epsilon\gamma_\lambda)\frac{i}{\not k}(ig\mu^\epsilon\gamma_\nu)\left(\frac{-ig^{\nu\sigma}}{(p-k)^2}\right)\left(\frac{-ig^{\rho\lambda}}{(p'-k)^2}\right)\times \\
&\quad (-ig\mu^\epsilon)\left[g_{\sigma\rho}(-(p-k)-(p'-k))_\mu + g_{\mu\rho}((p'-k)+(p'-p))_\sigma + g_{\mu\sigma}(-(p'-p)+(p-k))_\rho\right]
\end{aligned}
$$

$$
\begin{aligned}
&\to \quad g^3 \int \frac{d^n k}{(2\pi)^n} \gamma_\lambda \gamma^\alpha \gamma_\nu \frac{k_\alpha}{(k^2)^3} \left[g^{\nu\lambda} 2k_\mu - g^\lambda_\mu k^\nu - g^\nu_\mu k^\lambda \right] \quad \text{keeping only the most singular stuff} \\
&\to \quad \tfrac{1}{4} g^3 \int \frac{d^n k}{(2\pi)^n} \gamma_\lambda \gamma^\alpha \gamma_\nu \frac{1}{(k^2)^2} \left[g^{\nu\lambda} 2g_{\mu\alpha} - g^\lambda_\mu g^\nu_\alpha - g^\nu_\mu g^\lambda_\alpha \right] \quad \text{using } k_\alpha k_\beta \to k^2 g_{\alpha\beta}/n \text{ with } n \to 4 \\
&\to \quad \tfrac{1}{4} g^3 \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2)^2} \left[-12\gamma_\mu \right] \quad \text{using } \gamma \text{ matrix identities in } 4\text{-dimensions} \\
&\to \quad -3g^3 \gamma_\mu \frac{i}{(4\pi)^2} \Gamma(2 - \frac{n}{2}) \quad \text{using } n = 4 \text{ everywhere except in singular } \Gamma \text{ function} \\
&\to \quad -3g^3 (-\frac{1}{2} C_A) \gamma_\mu \frac{i}{(4\pi)^2 \epsilon} \quad \text{after bringing in the color factor we computed earlier}
\end{aligned}
\tag{451}
$$

So, now let us assemble the $\frac{1}{\epsilon}$ pieces, neglecting all finite corrections,

$$
ig\Lambda_\mu = ig \frac{g^2}{(4\pi)^2} \frac{1}{\epsilon} \gamma_\mu \left(C_F - \frac{1}{2} C_A + \frac{3}{2} C_A \right) = ig \frac{g^2}{(4\pi)^2} \frac{1}{\epsilon} \gamma_\mu (C_F + C_A) \,, \tag{452}
$$

to be multiplied (in the convention we are using) by the color matrix L^a. We must counter this $\frac{1}{\epsilon}$ piece by a counter term. We introduce (taking into account the basic sign of the gluon-quark-antiquark interaction, $+g\overline{\psi} \not{A} \psi$,

$\not{A} = \gamma^\mu A^a_\mu L^a$)

$$\Delta\mathcal{L} = Dg\overline{\psi}\not{A}\psi \tag{453}$$

which yields Feynman rule $iDg\gamma_\mu L^a$ from which we see that we require

$$D = -\frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}(C_F + C_A)\,. \tag{454}$$

The net form of the Lagrangian is then

$$\mathcal{L} + \Delta\mathcal{L} \ni (1+D)g\overline{\psi}\not{A}\psi \equiv Z_1 g\overline{\psi}\not{A}\psi \tag{455}$$

gives

$$Z_1 = 1 - \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}(C_F + C_A)\,. \tag{456}$$

We also recall our earlier results of

$$Z_2 = 1 - \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}C_F \tag{457}$$

and

$$Z_3 = 1 + \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\left(\frac{5}{3}C_A - \frac{4}{3}(\frac{1}{2}n_f)\right)\,. \tag{458}$$

The final stage is to relate the bare coupling to the renormalized coupling:

$$\begin{aligned} g_0 \overline{\psi}_0 \not{A}_0 \psi_0 &= g\mu^\epsilon Z_1 \overline{\psi} \not{A} \psi \\ &= g\mu^\epsilon \frac{Z_1}{Z_2\sqrt{Z_3}} \overline{\psi}_0 \not{A}_0 \psi_0 \end{aligned} \tag{459}$$

from which we have

$$\begin{aligned} g_0 = g\mu^\epsilon \frac{Z_1}{Z_2\sqrt{Z_3}} &= g\mu^\epsilon \left(1 - \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\left[(C_F + C_A) - C_F + \frac{1}{2}\left(\frac{5}{3}C_A - \frac{4}{3}(\frac{1}{2}n_f)\right)\right]\right) \\ &= g\mu^\epsilon \left(1 - \frac{1}{2}\frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\left[\frac{11}{3}C_A - \frac{4}{3}(\frac{1}{2}n_f)\right]\right) . \end{aligned} \tag{460}$$

Taking the $\mu\frac{\partial g_0}{\partial \mu}$ derivative gives us

$$0 = \epsilon g\mu^\epsilon \left(1 - \frac{1}{2}\frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\left[\frac{11}{3}C_A - \frac{4}{3}(\frac{1}{2}n_f)\right]\right) + \mu^\epsilon \mu\frac{\partial g}{\partial \mu}\left(1 - \frac{3}{2}\frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\left[\frac{11}{3}C_A - \frac{4}{3}(\frac{1}{2}n_f)\right]\right) \tag{461}$$

leading to (after dividing through, expanding the denominator, and taking $\epsilon \to 0$)

$$\mu\frac{\partial g}{\partial \mu} \equiv \beta(g) = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C_A - \frac{4}{3}(\frac{1}{2}n_f)\right] . \tag{462}$$

For $C_A = 3$ we have

$$\mu \frac{\partial g}{\partial \mu} \equiv \beta(g) = -\frac{g^3}{(4\pi)^2}\left[11 - \frac{2}{3} n_f\right], \qquad (463)$$

which is indeed negative so long as $n_f < 33/2$. We have 2 colored fermions per family and 3 families, for a total of 6 flavors that we know of. So, we are not very close to 17. It is actually a closer call when we go to supersymmetry. But that is the subject of another course.

Of course, Eq. (463) can be integrated. For the moment, we write

$$\mu \frac{\partial g}{\partial \mu} = -\frac{b}{16\pi^2} g^3 \quad \Rightarrow \frac{dg}{g^3} = -\frac{b}{16\pi^2}\frac{d\mu}{\mu} \Rightarrow dg^{-2} = \frac{b}{8\pi^2} d\ln\mu \qquad (464)$$

which integrates to (g_0 refers to an initial value at μ_0)

$$\frac{1}{g^2(\mu)} - \frac{1}{g_0^2} = \frac{b}{8\pi^2}\ln\frac{\mu}{\mu_0} \qquad (465)$$

or, equivalently,

$$g^2(\mu) = \frac{g_0^2}{1 + \frac{b}{8\pi^2} g_0^2 \ln\frac{\mu}{\mu_0}} \qquad (466)$$

or, equivalently, defining $\alpha_s = \frac{g^2}{4\pi}$

$$\alpha_s(\mu) = \frac{\alpha_0}{1 + \frac{b}{2\pi}\alpha_0 \ln\frac{\mu}{\mu_0}} \,. \tag{467}$$

As in the QED case, we can define a Λ such that the above equation can be rewritten as

$$\alpha_s(\mu) = \frac{1}{\frac{b}{2\pi}\ln(\mu/\Lambda)} = \frac{4\pi}{(11 - 2n_f/3)\ln(\mu^2/\Lambda^2)} \,. \tag{468}$$

For $11 - 2n_f/3 > 0$ (as we believe), α_s decreases with increasing μ^2.

Grand Unification in $SU(5)$

First, we must endure a few more technical details.

Two-component fermion results vs. Dirac four-component

The first of these is to simply note that the computation that we did of the fermion-loop contribution to $\Pi_{\mu\nu}$ implicitly employed a full Dirac fermion loop with coupling to the gauge boson current given by $ig\mu^\epsilon\gamma_\mu$ — i.e. no helicity restriction was imposed at the vertex. For is about to come, we need to note the alteration that occurs if we deal only with a right- or left-handed fermion, with coupling $-ie\mu^\epsilon\gamma_\mu\frac{1}{2}(1\pm\gamma_5)$. Referring back to Eq. (299) but inserting the helicity projected vertices above, we would have had (remembering the minus sign for a closed fermion loop and shifting to $p'=p-\alpha k$ at a certain stage and assuming a left-handed fermion):

$$
\begin{aligned}
i\Pi_{\mu\nu} &= -(-ie\mu^\epsilon)^2\int\frac{d^np}{(2\pi)^n}\mathrm{Tr}\left(\gamma_\mu\frac{1}{2}(1-\gamma_5)\frac{i}{\not p-m}\gamma_\nu\frac{1}{2}(1-\gamma_5)\frac{i}{\not p-\not k-m}\right)\\
&= -e^2\mu^{2\epsilon}\int\frac{d^np}{(2\pi)^n}\frac{\mathrm{Tr}\left[\gamma_\mu\frac{1}{2}(1-\gamma_5)(\not p+m)\gamma_\nu\frac{1}{2}(1-\gamma_5)(\not p-\not k+m)\right]}{(p^2-m^2)[(p-k)^2-m^2]}\\
&= -e^2\mu^{2\epsilon}\int d\alpha\int\frac{d^np'}{(2\pi)^n}\frac{\mathrm{Tr}\left[\gamma_\mu\frac{1}{2}(1-\gamma_5)(\not p'+\alpha\not k+m)\gamma_\nu\frac{1}{2}(1-\gamma_5)(\not p'-(1-\alpha)\not k+m)\right]}{[p'^2-m^2+k^2\alpha(1-\alpha)]^2}
\end{aligned}
$$

$$= \quad -e^2\mu^{2\epsilon}\int d\alpha \int \frac{d^n p'}{(2\pi)^n} \frac{\mathrm{Tr}\left[\gamma_\mu \frac{1}{2}(1-\gamma_5)(\not p' + \alpha \not k)\gamma_\nu(\not p' - (1-\alpha)\not k)\right]}{[p'^2 - m^2 + k^2\alpha(1-\alpha)]^2} \tag{469}$$

At this point, let us work on simplifying the numerator. We must remember that terms that are odd in p' will integrate to 0. Further, we must recall that the trace of an odd number of γ matrices is 0. Finally, the trace involving the γ_5 is 0 for the terms even in p' because of the appearance of either two $\not p'$'s or two $\not k$'s. Using these ingredients, we obtain:

$$\begin{aligned} N &= \frac{1}{2}[p'^\rho p'^\sigma - k^\rho k^\sigma \alpha(1-\alpha)]\mathrm{Tr}[\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma] \\ &= \frac{1}{2}[p'^\rho p'^\sigma - k^\rho k^\sigma \alpha(1-\alpha)]2^{n/2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma} + g_{\mu\sigma}g_{\nu\rho})] \,. \end{aligned} \tag{470}$$

If we neglect the m^2 term in our earlier Eq. (300) (to get the m^2 terms correct, we would have to get involved with helicity flip diagrams — this is unnecessary for the important point below) then this is the same result as in our earlier calculation, but with one difference — the extra factor of $\frac{1}{2}$.

This $\frac{1}{2}$ carries through everything and so for a fermion *of given helicity* we get just half of our earlier contribution to Z_3. As a result, the β function contribution is $\frac{1}{2}$ of what we had for a 4-component Dirac particle. For each

2-component fermion of definite helicity we get

$$\frac{ie^2}{24\pi^2}\frac{1}{\epsilon}\left[p_\mu p_\nu - p^2 g_{\mu\nu}\right] = \frac{ie^2}{16\pi^2}\left(\frac{2}{3}\right)\frac{1}{\epsilon}\left[p_\mu p_\nu - p^2 g_{\mu\nu}\right] . \tag{471}$$

A spin-zero loop — e.g. a Higgs boson loop

Imagine that we have a charged Higgs boson in QED, with charge 1, i.e. the same absolute charge as that of the electron. In order to include its effect in the running of e, we need to compute its impact on Z_3 through its impact on $\Pi_{\mu\nu}$.[4]

So, first recall the Feynman rules for a charged spin-zero particle coupling to the photon. These are generated from the Lagrangian form

$$[(\partial^\mu - ieA^\mu)\phi]^\dagger[(\partial_\mu - ieA_\mu)\phi] \ni -ieA_\mu[\partial^\mu\phi^\dagger]\phi + ieA^\mu\phi^\dagger\partial_\mu\phi + e^2A^\mu A_\mu\phi^\dagger\phi . \tag{472}$$

(Note the opposite sign of the $-ieA_\mu$ in the covariant derivative compared to our convention for the electron. This is because I am discussing a positively charged Higgs vs. the negatively charged electron.)

[4] One-loop diagrams for the fermion $-i\Sigma$ and the photon-electron-electron vertex, Λ_μ, in which the charged Higgs boson is emitted by the electron, leaving behind a neutrino, and then the neutrino reabsorbs the charged Higgs can be neglected in the small m limit — Higgs bosons couple proportional to mass. Thus, only $\Pi_{\mu\nu}$ needs to be computed.

These generate two Feynman rules. The first is one generated by the first two terms above and is one in which the H^+ enters from the left with momentum p, absorbs a photon with momentum q and Lorentz index μ and exits to the right with momentum $p' = p + q$. The Feynman rule for this vertex is

$$ie(p' + p)_\mu \,. \tag{473}$$

The second diagram is generated by the last term above. It is a 4-particle interaction involving two photons with Lorentz indices μ and ν connected to an incoming H^+ and and outgoing H^+. There are two contractions of the two external photons with the $A^\mu A_\mu$ fields yielding a factor of 2. The resulting Feynman rule is

$$2ie^2 g_{\mu\nu} \,. \tag{474}$$

With these Feynman rules, we find there are two one-charged-Higgs-loop diagrams contributing to $\Pi_{\mu\nu}$. These are: a) photon to charged Higgs pair (H^+ with k entering vertex μ and exiting with $p + k$), following by charged Higgs pair recombining at vertex ν; and b) charged Higgs "tadpole" correction to the photon propagator, with tadpole loop momentum k. The expressions for these are

- **a)**

$$(ie\mu^{\epsilon})^2 \int \frac{d^n k}{(2\pi)^n} \frac{i}{k^2 - m^2} (p + 2k)_\mu \frac{i}{(p+k)^2 - m^2} (p + 2k)_\nu \,; \qquad (475)$$

- **b)**

$$2ie^2 g_{\mu\nu} \int \frac{d^n k}{(2\pi)^n} \frac{i}{k^2 - m^2} \,. \qquad (476)$$

These we evaluate using standard techniques.

$$\begin{aligned}
a) \quad &= (ie\mu^{\epsilon})^2 \int \frac{d^n k}{(2\pi)^n} \frac{i}{k^2 - m^2} (p + 2k)_\mu \frac{i}{(p+k)^2 - m^2} (p + 2k)_\nu \\
&= (e\mu^{\epsilon})^2 \int d\alpha \int \frac{d^n k}{(2\pi)^n} \frac{(p + 2k)_\mu (p + 2k)_\nu}{(k^2 + 2\alpha k \cdot p + \alpha p^2 - m^2)^2} \\
&= (e\mu^{\epsilon})^2 \int d\alpha \int \frac{d^n k'}{(2\pi)^n} \frac{((1 - 2\alpha)p + 2k')_\mu ((1 - 2\alpha)p + 2k')_\nu}{(k'^{\,2} + \alpha(1 - \alpha)p^2 - m^2)^2} \\
&= (e\mu^{\epsilon})^2 \int d\alpha \int \frac{d^n k'}{(2\pi)^n} \frac{(1 - 2\alpha)^2 p_\mu p_\nu + 4k'_\mu k'_\nu}{(k'^{\,2} + \alpha(1 - \alpha)p^2 - m^2)^2} \\
&\overset{n \to 4}{=} e^2 \frac{i}{(4\pi)^2} \int d\alpha \left[(1 - 2\alpha)^2 p_\mu p_\nu \Gamma(2 - \frac{n}{2}) + 4(\frac{1}{2} g_{\mu\nu})[\alpha(1 - \alpha)p^2 - m^2]\Gamma(1 - \frac{n}{2}) \right] \\
&= \frac{ie^2}{(4\pi)^2} \left[\frac{1}{3} p_\mu p_\nu \Gamma(2 - \frac{n}{2}) + (\frac{1}{3} p^2 - 2m^2) g_{\mu\nu} \Gamma(1 - \frac{n}{2}) \right]
\end{aligned}$$

$$
\begin{aligned}
b) \quad &= \quad 2i(e\mu^{\epsilon})^2 g_{\mu\nu} \int \frac{d^n k}{(2\pi)^n} \frac{i}{k^2 - m^2} \\
&= \quad 2i(e\mu^{\epsilon})^2 g_{\mu\nu} \int \frac{d^n k}{(2\pi)^n} \frac{i}{k^2 - m^2} \frac{(p+k)^2 - m^2}{(p+k)^2 - m^2} \\
&= \quad 2i(e\mu^{\epsilon})^2 g_{\mu\nu} i \int d\alpha \int \frac{d^n k'}{(2\pi)^n} \frac{(1-\alpha)^2 p^2 + k'^2 - m^2}{[k'^2 + \alpha(1-\alpha)p^2 - m^2]^2} \\
&= \quad 2i(e\mu^{\epsilon})^2 g_{\mu\nu} i \int d\alpha \int \frac{d^n k'}{(2\pi)^n} \left[\frac{[(1-\alpha)^2 - \alpha(1-\alpha)]p^2}{[k'^2 + \alpha(1-\alpha)p^2 - m^2]^2} + \frac{1}{k'^2 + \alpha(1-\alpha)p^2 - m^2} \right] \\
&\overset{n\to 4}{=} \quad -2e^2 g_{\mu\nu} \frac{i}{(4\pi)^2} \left[\frac{1}{6} p^2 \Gamma(2 - \frac{n}{2}) + \left(\frac{1}{6} p^2 - m^2 \right) \Gamma(1 - \frac{n}{2}) \right] \\
a) + b) \quad &= \quad \frac{ie^2}{(4\pi)^2} \left[\tfrac{1}{3} p_\mu p_\nu \Gamma(2 - \frac{n}{2}) - \tfrac{1}{3} p^2 g_{\mu\nu} \Gamma(2 - \frac{n}{2}) \right] \\
&\overset{n\to 4}{=} \quad \frac{ie^2}{48\pi^2} \frac{1}{\epsilon} \left[p_\mu p_\nu - p^2 g_{\mu\nu} \right] . \qquad (477)
\end{aligned}
$$

This should be compared to the old result for the 4-component Dirac electron which was

$$
\frac{ie^2}{12\pi^2} \frac{1}{\epsilon} \left[p_\mu p_\nu - p^2 g_{\mu\nu} \right] . \qquad (478)
$$

The rule is: a charged spin-0 loop gives $1/4$ the result of a Dirac fermion (i.e. including both electron helicities) or $1/2$ the result for a fermion of definite helicity.

In the above, you will see I used the same sort of trick as previously

for gauge boson loop corrections to a gauge propagator to cancel away the $\Gamma(1-\frac{n}{2})$ that cannot actually be present. (There should be no quadratically divergent contribution to $\Pi_{\mu\nu}$ because of gauge invariance requirements: the Ward identity $p^\mu \Pi_{\mu\nu} = 0$ should hold in an arbitrary number of dimensions.)

Putting it all together

So let us summarize where we are.

1. For $SU(3)$ we have

$$\beta_3(g_3) = -\frac{g_3^3}{16\pi^2}\left[\frac{11}{3}3 - \frac{2}{3}\left(\frac{1}{2}\right) n_{triplets\ of\ given\ helicity}\right], \tag{479}$$

where we have introduced the $\frac{1}{2}$ in the last term appropriate when counting fundamental representations of given helicity. We have also introduced the notation g_3 for the $SU(3)$ strong coupling constant which we had been calling g.

Each generation or family has u_L, d_L, u_R, d_R (L, R denote helicity), each coming with three "colors" of the fundamental $SU(3)$ triplet. Thus,

$n_{triplets\ of\ given\ helicity} = 4N_g$, and we obtain

$$\beta_3(g_3) = -\frac{g_3^3}{16\pi^2}\left[11 - \frac{4}{3}N_g\right] . \qquad (480)$$

Note that the Higgs boson does not have "color" (does not interact strongly) and so is not counted above.

2. For the weak $SU(2)$ we have

$$\beta_2(g_2) = -\frac{g_2^3}{16\pi^2}\left[\frac{11}{3}2 - \frac{2}{3}\left(\frac{1}{2}\right) n_{weak-isospin\ doublets\ of\ given\ helicity}\right] , \qquad (481)$$

What weak-isospin doublets do we have in the SM? In each family or generation we have: $\left(\begin{array}{c} u \\ d \end{array}\right)_L$ in three colors and $\left(\begin{array}{c} \nu_e \\ e^- \end{array}\right)_L$ (uncolored). The right-handed quarks and right-handed electron are weak-isospin singlets (i.e. are not seen by the $SU(2)$ gauge bosons) and do not contribute.

Thus, we have $n_{weak-isospin\ doublets\ of\ given\ helicity} = 4N_g$ leading to

$$\beta_2(g_2) = -\frac{g^3}{16\pi^2}\left[\frac{22}{3} - \frac{4}{3}N_g\right] \tag{482}$$

from the family members. But, we are not done! In the SM, we have a Higgs doublet: $\begin{pmatrix} H^+ \\ H^0 \end{pmatrix}$. It counts $\frac{1}{2}$ as much as a fermion doublet of given helicity. (This is a slight generalization of the QED calculation we did under the technical notes precursor discussion.) (Note, I will do the counting ignoring electroweak symmetry breaking — the β functions obtained are those appropriate above the EWSB scale of $v \sim 246$ GeV.)

Adding this guy in (I actually allow for an arbitrary number of such Higgs doublets) gives:

$$\beta_2(g_2) = -\frac{g_2^3}{16\pi^2}\left[\frac{22}{3} - \frac{4}{3}N_g - \frac{1}{6}N_{D_1}\right] \tag{483}$$

where D means doublet and the 1 subscript indicates the $Y = 1$ hypercharge needed for the $Q = T_3 + \frac{1}{2}Y$ operator to give correct charges to the two components of the Higgs doublet.

3. **Finally, we have the SM $U(1)$ group. I will temporarily keep the notation g' for the coupling. For reasons to be explained, $g' = -\sqrt{\frac{3}{5}}g_1$. For this group, the well-known coupling is $g'\frac{Y}{2}$. This is the charge that replaces the QED charge e. We must count all particles that can give a contribution to Z_3. The list is extensive. In the case of fermions, we count separately different helicities and remember that we have to reduce the 4-component QED result by a factor of $\frac{1}{2}$.**

$$
\begin{array}{ccc}
\text{particle} & Y & \left(\frac{Y}{2}\right)^2 \\
u_L & \frac{1}{3} & \frac{1}{36} \\
d_L & \frac{1}{3} & \frac{1}{36} \\
u_R & \frac{4}{3} & \frac{16}{36} \\
d_R & -\frac{2}{3} & \frac{4}{36} \\
e_L^- & -1 & \frac{1}{4} \\
\nu_L & -1 & \frac{1}{4} \\
e_R^- & -2 & 1 \\
\hline
\text{Sum} & = & \frac{10}{3}
\end{array}
\qquad (484)
$$

This is what we get for each of the N_g generations.

For the Higgs doublet, we get

$$\begin{array}{ccc} \text{particle} & Y & \left(\frac{Y}{2}\right)^2 \\ H^+ & 1 & \frac{1}{4} \\ H^0 & 1 & \frac{1}{4} \\ \hline \text{Sum} & = & \frac{1}{2} \end{array} \tag{485}$$

Here, we have to use the spin-0 calculation which showed that a scalar particle comes in with $1/2$ the weight of a fermion of given helicity.

The result is

$$\beta(g') = \frac{g'^{\,3}}{24\pi^2}\left[\frac{10}{3}N_g + \frac{1}{2}\left(\frac{1}{2}\right)N_{D_1}\right] . \tag{486}$$

If we now throw in the $g' = -\sqrt{\frac{3}{5}}g_1$ factor, the effective $\beta_1(g_1)$ is

$$\beta_1(g_1) = \frac{g_1^3}{16\pi^2}\left[\frac{4}{3}N_g + \frac{1}{10}N_{D_1}\right] . \tag{487}$$

(Recall that $\beta(g)$ is defined by $\mu\frac{\partial g}{\partial\mu} = \beta(g)$ for a general coupling g. Thus, if we define $g = f\hat{g}$, since $\beta(g) = Kg^3$, the β function for $\hat{g}$ only picks up a factor of f^2: $\beta(\hat{g}) = Kf^2\hat{g}^3$. In our case $f^2 = 3/5$.)

The only question is why is it appropriate to consider g_1 (when discussing coupling constant unification) as opposed to g'? This has to do with the idea that a group such as $SU(5)$ is the appropriate unification group.

$SU(5)$

The idea of grand unification is that all of our fundamental forces should actually be manifestations of a single force. In the QFT language, this means that there should be a single group that contains subgroups corresponding to the subgroups required for the forces we observe: i.e. we need a master non-abelian group that contains $SU(3) \times SU(2) \times U(1)$. This means that the master group should be large enough to contain the 4 commuting generators of the SM — 2 from color $SU(3)$ (denoting colors by $1, 2, 3 = r, g, b$ the diagonal traceless ones could be chosen as $\frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$ and $\frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$) and 1 for the $SU(2)$ T^3 generator associated with the W^3 field and 1 for the $U(1)$ generator associated with the B field. The generator for the $U(1)$ group must be chosen so that it commutes with the $SU(3)$ and $SU(2)$ representation matrices.

The simplest such group is $SU(5)$. $SU(5)$ contains 24 generators associated with $\mathbf{24}$ fields, which we might denote by A^i_j. They belong to the adjoint representation, which is a $\mathbf{24}$ and decomposes to $SU(3)$ and $SU(2)$ representations as

$$\mathbf{24} = (\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{2}) + (\bar{\mathbf{3}},\mathbf{2}). \tag{488}$$

Considering the generators in a 5×5 traceless matrix (fundamental) representation, we would associate:

1. the upper 3×3 sub-block matrix with $SU(3)$ (A^a_b, $a,b = 1,2,3$, associated with the gluon fields G^a_b – more or less, as we shall explain) — this would be the $(8,1)$ part of the 24;

2. the lower 2×2 sub-block with $SU(2)$ (A^r_s, $r,s = 4,5$, which can be identified with the $W^{\pm}, W^3$ fields of the SM) — this would be the $(1,3)$ part of the 24;

3. the remaining orthogonal traceless composition of the diagonal matrices that commute with all upper 3×3 matrices and all lower 2×2 matrices with the $U(1)$ generator — this would be the $(1,1)$ part of the 24.

This means that the $U(1)$ generator should be a linear combination of

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{489}$$

and

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{490}$$

The appropriate matrix, T^0, normalized so that $\mathrm{Tr}[T^0 T^0] = \frac{1}{2}$ (our

convention) is

$$T^0 = \frac{1}{2}\lambda^0 = \frac{1}{2\sqrt{15}} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} . \tag{491}$$

The corresponding T^3 generator associated with the W^3 is

$$T^3 = \frac{1}{2}\lambda^3 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} . \tag{492}$$

The field corresponding to the B field is then written in terms of the A^i_j fields in the form:

$$B = -\frac{1}{\sqrt{15}} A^\alpha_\alpha + \sqrt{\frac{3}{20}} A^r_r \,, \tag{493}$$

where the overall normalization is chosen so that the kinetic term for B would have canonical normalization ($\frac{1}{2}\partial^\mu B \partial_\mu B$) provided we started with canonical normalization for the kinetic term for the A^i_j fields: $\frac{3}{20} \times 2 + \frac{1}{15} \times 3 = \frac{1}{2}$. An overall matrix representation of all this is:

$$A \equiv \sum_{a=0}^{a=23} A^a \frac{\lambda^a}{2} = \frac{1}{\sqrt{2}} \begin{pmatrix} \left(G^\alpha_\beta - \frac{2}{\sqrt{30}} \delta^\alpha_\beta B \right) & & & X_1 & Y_1 \\ & & & X_2 & Y_2 \\ & & & X_3 & Y_3 \\ X^1 & X^2 & X^3 & \frac{1}{\sqrt{2}} W^3 + \frac{3}{\sqrt{30}} B & W^+ \\ Y^1 & Y^2 & Y^3 & W^- & -\frac{1}{\sqrt{2}} W^3 + \frac{3}{\sqrt{30}} B \end{pmatrix} . \tag{494}$$

In the above, the extra gauge bosons denoted by X_i and Y_i, associated with the $(3,2)$ part of the 24 (and their conjugates X^i and Y^i, associated with the $(\overline{3},2)$ part of the 24) must be very heavy since they would yield transitions that would lead to proton decay, something that can only occur at a very suppressed level. Presumably, their masses are of order the M_U scale at which this master $SU(5)$ group is broken down to $SU(3) \times SU(2) \times U(1)$. We do not have time to go into details about how this breaking occurs. Usually, extra Higgs fields (in particular a 24 Higgs field) are introduced to accomplish this breaking in a manner analogous to that you have studied for breaking $SU(2) \times U(1)$ down to $U(1)_{EM}$.

So, now the key point related to the matrix T^0 and the relation between g' and g_1 is to note that the covariant derivative for the $SU(5)$ group will take the form (prior to breaking down to $SU(3) \times SU(2) \times U(1)$)

$$\partial_\mu - i g_5 A_\mu \tag{495}$$

where

$$A_\mu = \sum_{a=0}^{a=23} A_\mu^a \frac{\lambda^a}{2} = \sum_{a=0}^{a=23} A_\mu^a T^a \tag{496}$$

with the T^a being the 5×5 matrices normalized according to our convention ($\mathrm{Tr}[T^a T^b] = \frac{1}{2}\delta^{ab}$) when we consider this covariant derivative operating on a fundamental 5-dimensional representation of $SU(5)$. Thus, if the groups all unify at M_U to a common group, we will have $g_1 = g_2 = g_3 = g_5$ at M_U. Below M_U, the coupling constants will diverge according to the individual $\beta_i(g_i)$, but we must use unification of, in particular, g_1 (and not g') with g_2 and g_3 as our starting point.

Now, we must fit the fermions into representations of $SU(5)$. This is done by putting part of them into a 5 (or, equivalently, $\bar{5}$) and part into a 10 (antisymmetric) representation. For a single family, we have to accommodate

$$
\begin{aligned}
(\nu_e, e^-)_L : &\qquad (1,2) \\
e^-_R = e^+_L : &\qquad (1,1) \\
(u_a, d_a)_L : &\qquad (3,2) \\
(u_a)_R = \bar{u}^a_L : &\qquad (\bar{3},1) \\
(d_a)_R = \bar{d}^a_L : &\qquad (\bar{3},1)
\end{aligned} \tag{497}
$$

where I have used the equivalence of a f_R state with the $\bar{f}_L$ state (under charge conjugation). Placement of these 1st family members is accomplished in the form

$$
\bar{5} = (\bar{d}^1, \bar{d}^2, \bar{d}^3, e^-, -\nu_e)_L \,, \tag{498}
$$

which could also be written as

$$
5 = (d_1, d_2, d_3, e^+ - \bar{\nu}_e)_R \,, \tag{499}
$$

and

$$10 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \bar{u}^3 & -\bar{u}^2 & u_1 & d_1 \\ -\bar{u}^3 & 0 & \bar{u}^1 & u_2 & d_2 \\ \bar{u}^2 & -\bar{u}^1 & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 e^+ & \\ -d_1 & -d_2 & -d_3 & -e^+ & 0 \end{pmatrix} . \tag{500}$$

In the above, the $1, 2, 3$ indices are the three $SU(3)$ indices in the fundamental representation, which we have earlier called r, g, b.

I have no time to go into details about the 10 representation. What we want to focus on is the fundamental 5 representation. This is the key to establishing the relation between the Y operator and the T^0 generator of the $U(1)$ subgroup of $SU(5)$. We know from our SM work that the hypercharge assignments of the 5 members are

$$Y[5] = diag[-2/3, -2/3, -2/3, 1, 1][5] . \tag{501}$$

Meanwhile,

$$T^0[5] = \frac{1}{2\sqrt{15}} diag[2, 2, 2, -3, -3][5] . \tag{502}$$

Thus, if we wish to make the identification

$$g_1 T^0 = g' \frac{1}{2} Y \,, \tag{503}$$

where g_1 will unify with g_2 and g_3 at scale M_U so that all take the value g_5 of the universal $SU(5)$ coupling constant at that scale, we must have

$$g_1 \frac{1}{2\sqrt{15}} \times 2 = g' \frac{1}{2} \times \left(-\frac{2}{3}\right) \tag{504}$$

or

$$g' = -\sqrt{\frac{3}{5}} g_1 \,, \tag{505}$$

as employed earlier in getting the form of β_1.

There are many other fascinating aspects to $SU(5)$ (and its generalizations), not the least of which is that it leads to the observed relation between quark and lepton charges. Since the charge operator Q must commute with all the color operators, it can only be some linear combination of T^3 and T^0. In fact, from the discussion just above, we see that

$$Q = T^3 + \frac{Y}{2} = T^3 - \sqrt{\frac{5}{3}} T^0 \tag{506}$$

so that for the fundamental $\mathbf{5}$ representation we have

$$Q[5] = diag[-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 1, 0][5]\,. \tag{507}$$

We could have multiplied by an arbitrary overall factor, but, once we have defined the e^+ charge to be $+1$ in charge units, the quarks must have the phenomenologically required fractional values!

Coupling Constant Unification

To review, we have:

$$\beta_1(g_1) = \frac{g_1^3}{16\pi^2}\left[\frac{4}{3}N_g + \frac{1}{10}N_{D_1}\right]\;; \tag{508}$$

$$\beta_2(g_2) = -\frac{g_2^3}{16\pi^2}\left[\frac{22}{3} - \frac{4}{3}N_g - \frac{1}{6}N_{D_1}\right]\;; \tag{509}$$

$$\beta_3(g_3) = -\frac{g_3^3}{16\pi^2}\left[11 - \frac{4}{3}N_g\right]\,. \tag{510}$$

The coupling evolution equations are then summarized in the form

$$\frac{\partial g_n}{\partial \ln \mu} = -\frac{b_n}{16\pi^2} g_n^3 \tag{511}$$

with

$$\begin{aligned} -b_1 &= \tfrac{4}{3}N_g + \tfrac{1}{5}(N_{S_2} + 4N_{S_4}) + \tfrac{1}{10}(N_{D_1} + 9N_{D_3}) + \tfrac{3}{5}N_{T_2} + \tfrac{28}{5}N_{34} \\ -b_2 &= \tfrac{4}{3}N_g + \tfrac{1}{6}(N_{D_1} + N_{D_3}) + \tfrac{2}{3}(N_{T_0} + N_{T_2}) + \tfrac{28}{27}N_{34} - \tfrac{22}{3} \\ -b_3 &= \tfrac{4}{3}N_g - 11\,. \end{aligned} \tag{512}$$

where I have introduced a bunch of other possible Higgs representations that might be present in a more general model: I denote the number of $|Y| = 1$ doublets by N_{D_1}, the number of $|Y| = 2$ triplets by N_{T_2}, and so forth; N_{34} denotes the number of $(I = 3, |Y| = 4)$ representations. I do not consider $|Y| \geq 6$ singlets, $|Y| \geq 5$ doublets, or $|Y| \geq 4$ triplets. Here, by doublets and triplets and so forth, I refer to the weak isospin representation.

In the Minimal Supersymmetric Model, one adds only the superpartners of the SM particles plus a 2nd Higgs doublet (as required for anomaly cancellation and for giving masses to both up and down type quarks). The resulting b_i are derived in the Appendix following this section. One finds:

$$-b_1 = 2N_g + \tfrac{3}{5}(N_{S_2} + 4N_{S_4}) + \tfrac{3}{10}(N_{D_1} + 9N_{D_3}) + \tfrac{9}{5}N_{T_2} + \tfrac{84}{5}N_{34}$$
$$-b_2 = 2N_g + \tfrac{1}{2}(N_{D_1} + N_{D_3}) + 2(N_{T_0} + N_{T_2}) + \tfrac{28}{9}N_{34} - 6$$
$$-b_3 = 2N_g - 9 \tag{513}$$

Note that in the above equations there is no influence from $Y = 0$ singlets, but $Y \neq 0$ singlets affect b_1. Note also that the N_g terms cancel in the difference of any two b_i's in both the SM and the MSSM. Finally, in the MSSM we note the upper limit of $N_g = 4$ in order that QCD be asymptotically free. (Phenomenological constraints on an extra generation are quite severe if one assumes coupling unification — basically, its members cannot be much heavier than the top quark of the 3rd generation.)

Now, the evolution equations are easily integrated to give

$$\frac{1}{g_n^2(\mu)} - \frac{1}{g_n^2(\mu_0)} = \frac{b_n}{8\pi^2} \ln\left(\frac{\mu}{\mu_0}\right) \tag{514}$$

or

$$\frac{1}{\alpha_n(\mu)} - \frac{1}{\alpha_n(\mu_0)} = \frac{b_n}{2\pi} \ln\left(\frac{\mu}{\mu_0}\right) . \tag{515}$$

We identify μ_0 with the unification scale M_U at which all the α_i are supposed to unify to a single value that we denote by $\alpha_5(M_U)$. The lower scale μ will be chosen to be of order the EWSB scale of order m_Z or $v \sim 246~\mathrm{GeV}$. By subtracting equations for two of the α_i assuming that all the α_i unify to α_5 at M_U we obtain two equations. A useful pair is

$$\begin{aligned} \frac{1}{\alpha_2} - \frac{1}{\alpha_3} &= \frac{b_2 - b_3}{2\pi} L \\ \frac{1}{\alpha_2} - \frac{1}{\alpha_1} &= \frac{b_2 - b_1}{2\pi} L \end{aligned} \tag{516}$$

where $L = \ln(\mu/M_U)$. We can solve the first equation for $L/2\pi$ and substitute, obtaining

$$\frac{1}{\alpha_2} - \frac{1}{\alpha_1} = \frac{b_2 - b_1}{b_2 - b_3} \left(\frac{1}{\alpha_2} - \frac{1}{\alpha_3} \right) . \tag{517}$$

We write this in the form:

$$\frac{1}{\alpha_2} \left(1 - \frac{b_2 - b_1}{b_2 - b_3} \right) = \frac{1}{\alpha_1} - \frac{1}{\alpha_3} \frac{b_2 - b_1}{b_2 - b_3} \tag{518}$$

or

$$\frac{1}{\alpha_2}(b_1 - b_3) - \frac{1}{\alpha_1}(b_2 - b_3) = \frac{1}{\alpha_3}(b_1 - b_2)\,. \tag{519}$$

At the EWSB scale, we have the phenomenological results that

$$\begin{aligned}
\alpha_1(\mu) &= \frac{5}{3}\frac{g'^2}{4\pi} = \frac{5}{3}\frac{\alpha_{QED}}{\cos^2\theta_W} \\
\alpha_2(\mu) &= \frac{g^2}{4\pi} = \frac{\alpha_{QED}}{\sin^2\theta_W} \\
\alpha_3(\mu) &= \alpha_s(\mu)\,.
\end{aligned} \tag{520}$$

We substitute these into the above equation. The requirement of unification then takes the form (with $\mu \sim m_Z$)

$$\alpha_s(m_Z) = \alpha_{QED}(m_Z)\frac{5(b_1 - b_2)}{\sin^2\theta_W(5b_1 + 3b_2 - 8b_3) - 3(b_2 - b_3)} \tag{521}$$

We emphasize that the above equation assumes "standard" $SU(5)$ normalization of the $U(1)$ coupling constant and a desert between m_Z and M_U. Once we have a solution to Eq. (521), we can solve for L and thence M_U using either of the two equations (516). We can then solve for $\alpha_5(M_U)$ using one of the equations (515).

Using $\alpha_s(m_Z) = 0.118$ and $\sin^2\theta_W = .2315$, and the results of Eqs. (512) and (513), the constraint of Eq. (521) reduces to

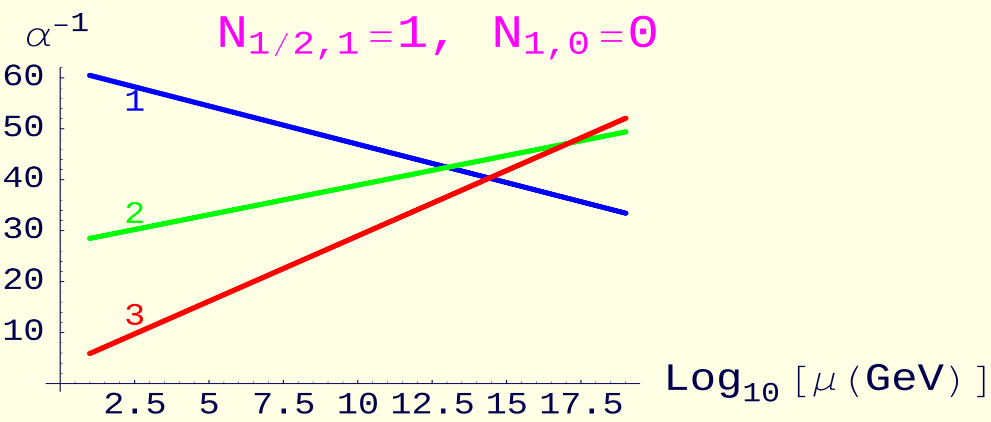
$$\begin{aligned} \mathrm{SM}: \quad 1 &\simeq -0.09 N_{S_2} - 0.36 N_{S_4} + 0.13 N_{D_1} - 0.22 N_{D_3} \\ &\quad + 0.71 N_{T_0} + 0.44 N_{T_2} - 1.39 N_{34} \end{aligned} \tag{522}$$

$$\begin{aligned} \mathrm{SUSY}: \quad 1 &\simeq -0.33 N_{S_2} - 1.31 N_{S_4} + 0.49 N_{D_1} - 0.82 N_{D_3} \\ &\quad + 2.61 N_{T_0} + 1.63 N_{T_2} - 5.11 N_{34} \end{aligned} \tag{523}$$

in the non-SUSY and MSSM cases, respectively. The exact coefficients in the equations above are sensitive to the precise α_s and $\sin^2\theta_W$ choices as well as to whether all the Higgs bosons (and SUSY sparticles) have mass near m_Z, as assumed, or nearer to 1 TeV. Two-loop corrections also lead to small changes in the unification conditions. Thus, the following discussion of 'solutions' should be regarded as being a somewhat rough, but indicative, guide to the possibilities.

First, we note that the simple SM with $N_{D_1} = 1$ and all others zero does not work, whereas the MSSM with (as required for anomaly cancellation)

$N_{D_1} = 2$ provides an almost exact solution. If you plot graphically, the SM failure looks less bad since the $\frac{1}{\alpha_i}$'s come close to crossing. Nonetheless, they do not. The figure below shows this lack of crossing. (In a change of notation, I write $N_{T,Y} =$ for the number of Higgs representations of a given type in the figure and in the subsequent discussion.)



In the SM case, we can only fix things up by bringing in the other Higgs representations (or other particles that we do not consider). We find that coupling unification can be achieved without SUSY by introducing additional Higgs representations in the standard model.

Some simple choices are:

Table 1: Higgs models that yield coupling unification in the SM context with α_s not far from 0.118

$N_{1/2,1}$	$N_{1/2,3}$	$N_{0,2}$	$N_{0,4}$	$N_{1,0}$	$N_{1,2}$	α_s	M_U (GeV)
1	0	0	2	0	0	0.106	4×10^{12}
1	0	4	0	0	1	0.112	7.7×10^{12}
1	0	0	0	0	2	0.120	1.6×10^{13}
2	0	0	0	1	0	0.116	1.7×10^{14}
2	0	2	0	0	2	0.116	4.9×10^{12}
2	1	0	0	0	2	0.112	1.7×10^{12}
3	0	0	0	0	1	0.105	1.2×10^{13}

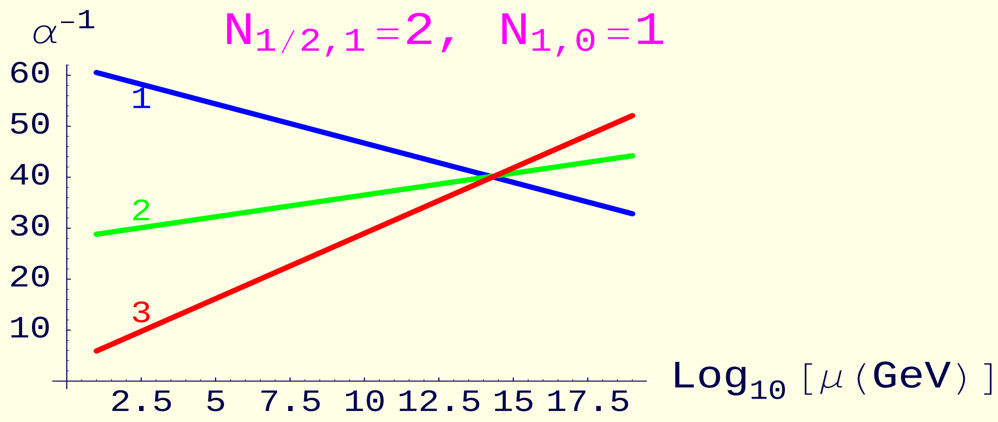
Remarks:

- **Find lower M_U than comfortable for proton decay.**

  Can fix by not having true group unification, as in some string models.

- **My personal favorite:** $N_{\frac{1}{2},1} = 2, N_{1,0} = 1 \Rightarrow \alpha_s(m_Z) = 0.115$, $M_U = 1.7 \times 10^{14}$ GeV

The unification of the couplings is displayed in the figure below.



This model has no phenomenological problems (other than needing the above fix for proton decay) so long as the neutral member of the $T = 1, Y = 0$ triplet has zero vacuum expectation value. In fact, this neutral triplet member is even a viable dark matter candidate.

Of course, without supersymmetry there is no cancellation of the quadratic divergences, implying that a light mass for the Higgs boson (as required by LEP data in the SM context) has no natural explanation. Something like supersymmetry is required to make a light Higgs boson natural.

Appendix: derivation of the MSSM b_i

β_3

It is simplest to begin with the b_3 additions due to the spin-0 partners of the spin-1/2 (two-component) quarks. In the first family, we have the partners $\widetilde{u}_L$, $\widetilde{u}_R$, $\widetilde{d}_L$, and $\widetilde{d}_R$. In total, there are four spin-0 fields per family, a total of $4N_g$ spin-0 fields. Since they are spin-0, they enter with only $1/2$ the weight of the two-component spin-1/2 fermions. Referring back to Eq (479), we get a correction to the bracket which then takes the form

$$[\ldots] \to \left[\ldots - \frac{2}{3}\left(\frac{1}{2}\right)\left(n_{fermionic\ triplets\ of\ given\ helicity} + \frac{1}{2} n_{bosonic\ triplets}\right)\right] . \tag{524}$$

We have

$$n_{fermionic\ triplets\ of\ given\ helicity} = n_{bosonic\ triplets} = 4N_g\,, \tag{525}$$

so that the bracket becomes

$$[\ldots - \frac{4}{3}N_g] \to [\ldots - 2N_g]\,. \tag{526}$$

But, we are not done. We also have fermionic partners to the gluons, the so-called gluinos. These are fermion partners of the gluons. There is one for each gluon. Let us recall that the contribution of a fermion to the standard $[\ldots]$ is

$$-\frac{2}{3}\times color\ factor\,, \tag{527}$$

where the $color\ factor = \frac{1}{2}$ for a fundamental triplet insertion into a gluon line is now replaced by the usual adjoint representation C_A which is 3 for $SU(3)$. The result in the case of β_3 is the replacement

$$\left[\frac{11}{3}3 - \ldots\right] \rightarrow \left[\left(\frac{11}{3} - \frac{2}{3}\right)3 - \ldots\right]\,, \tag{528}$$

implying that the previous $11 \rightarrow 9$ inside the $[\ldots]$.

$\boxed{\beta_2}$

Here, we get a modification of Eq. (481) in which

$$n_{weak-isospin\ doublets\ of\ given\ helicity} \rightarrow n_{weak-isospin\ doublets\ of\ given\ helicity} + \frac{1}{2}n_{weak-isospin\ bosonic\ doublets}\,. \tag{529}$$

Since there is a squark or slepton for each L or R helicity quark or lepton, we have

$$n_{weak-isospin\ bosonic\ doublets} = n_{weak-isospin\ doublets\ of\ given\ helicity} = 4N_g \qquad (530)$$

and the $[\ldots]$ in the form of $\beta_2(g_2)$ obtained before including the Higgs field gets modified in a manner very analogous to the color case:

$$\left[\ldots - \frac{4}{3}N_g\right]] \to \left[\ldots - \frac{4}{3}N_g\frac{3}{2}\right] = [\ldots - 2N_g]\ . \qquad (531)$$

Note that once again we are getting the same (now $-2N_g$) term in both β_3 and β_2.

Similar to β_3, we must include the effect on β_2 of the fermionic partners of the weak gauge bosons, the so-called "winos". By analogy, it should be clear that in the $[\ldots]$, we have

$$\left[\frac{11}{3}2 - \ldots\right] \to \left[\left(\frac{11}{3} - \frac{2}{3}\right)2 - \ldots\right] = [6 - \ldots]\ , \qquad (532)$$

where the 2 outside the parenthesis is just C_A for the $SU(2)$ group.

The Higgs fields require special discussion. First, there are now two Higgs complex bosonic doublet fields. Second, each of these fields comes with a fermionic partner which contributes twice as much. As a result, the $-\frac{1}{6}N_{D_1}$ component of β_2 is replaced by

$$-\frac{1}{6}N_{D_1} \to -\frac{1}{6}N_{D_1} \times 3 = -\frac{1}{2}N_{D_1}, \tag{533}$$

where, in addition, $N_{D_1} = 2, 4, 6, \ldots$ is required for anomaly cancellation.

$\boxed{\beta_1}$

This requires going through the listing of the earlier tables and multiplying each fermion contribution listed by $\frac{3}{2}$ and each boson contribution listed by 3. The result is

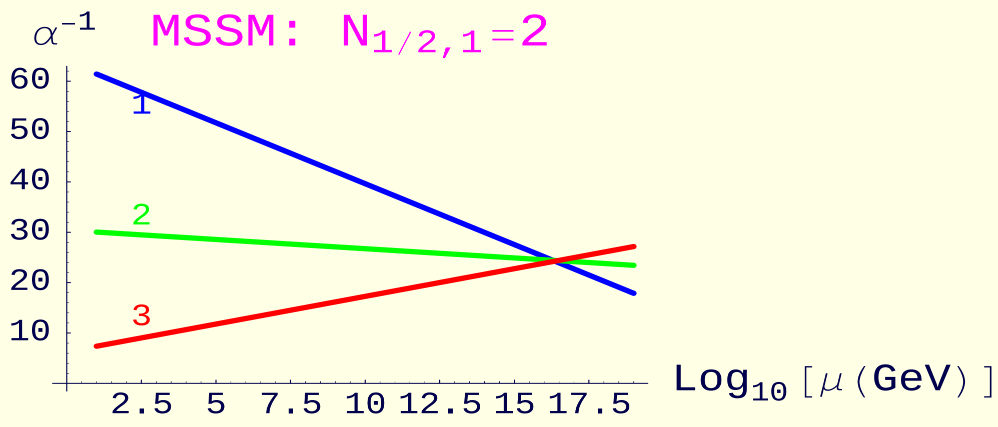
$$\left[\frac{4}{3}N_g + \frac{1}{10}N_{D_1}\right] \to \left[2N_g + \frac{3}{10}N_{D_1}\right]. \tag{534}$$

There is, of course, also a bino partner to the B, but since the group is $U(1)$ there is no B-bino-bino vertex and so no effect on b_1 from this particle.

You can now check that these modifications give the b_i values quoted for the MSSM.

Unification plots

Of course, it is interesting to plot the coupling unification for the MSSM to compare to the graphs obtained earlier for the SM.



Note the much higher scale of $M_U \sim 1.9 \times 10^{16}$ GeV for the unification point. This means that the MSSM is much safer (but still not entirely safe) with regard to proton if there is true group unification — the required X and Y bosons are much heavier than in the ad hoc SM case I gave as an example earlier.

Also note the large value of α_5 at the unification point.

Computation of $\Pi_{\mu\nu}$ to order g^2e^2

This I will do from my notes (Renormalization Notebook). The idea is to derive the first important correction appearing in the formula

$$R = \frac{\sum_q \sigma(e^+e^- \to q\overline{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_q e_q^2 \left(1 + \frac{\alpha_s(Q^2)}{\pi}\right) \quad (535)$$

using the $\Pi_{\mu\nu}$ computation.