Advanced Topics
in
Effective Field Theory

A Modular Course Offered at
The University of Toronto Department of Physics
Fall, 2008

by

Andrew E. Blechman

Dedication

I would like to dedicate these lecture notes to the memory of

Eun-Jung Rhee

my classmate, officemate, collaborator and friend.

Acknowledgments

These lectures come from a minicourse I taught at the University of Toronto in the fall semester of 2008. The students were mostly high energy theory students, along with a few cosmologists and condensed matter theory students. I am very grateful for their attendance, and hope that they got as much out of my lectures as I did, if not more. William Cheung, Santiago De Lope Amigo, Simon Freedman, Siavash Golkar, Alex Venditti, William Witczak-Krempa and Saba Zuberi.

I am also very grateful to Erich Poppitz and Michael Luke, who I would bother now and then to help in my organization of the course and to work out some finer points about anomalies and effective actions.

Syllabus

Instructor: Dr. Andrew Blechman (MP 1102, x8-5215, blechman@physics); supervised by Prof. Michael Luke (luke@physics).

Length: 4 weeks, two 1.5 hr lectures each week.

Proposed Time: Alternate weeks, Monday/Wednesday, 3PM - 5 PM in MP 1115.

Prerequisites: QFT I and II or equivalent, or permission of instructor.

Course Goals: There is a gap of knowledge between the QFT courses students take when they start graduate work in theoretical physics, and actual research. This gap is traditionally filled by hard work and hands-on experience. I would like to offer a short course to help students fill that gap. I will try to cover the basics of three vitally important techniques used in modern theoretical physics research: spontaneous symmetry breaking, effective field theory and the renormalization group. Materials will be covered in both lecture format, as well as through challenging homework problems.

Evaluations: Students will complete one assignment each week for a total of 4 assignments. Each set will contain several problems. All problems will be weighted equally and will be graded with the following scheme:

1. 0 points: did not attempt.
2. 1 point: did not put any effort into trying to get the answer.
3. 2 points: made an honest effort; almost got the answer.
4. 3 points: did the problem completely right.

The sum of all points earned will be the final raw grade. Allowances will be made for participation in and out of class. Students are strongly encouraged to work together on the assignments, as they will (hopefully) be challenging.

Contents

Chapter 1

Group Theory: The Language of Model Building

When trying to understand the physics involved in a problem, the first step is to identify the relevant degrees of freedom and any symmetries that relate them. Once you have done that, you will be in a position to think about how these degrees of freedom are allowed to interact with each other. In this lecture, we will consider how to identify these symmetries, and how to codify them. The mathematics that are used to accomplish this comes from group theory. This is not a group theory class, and I do not want to go deeply into the rich and fascinating structure of the theory of groups, but there are some things that we have to understand. After a brief review of the basics, we will discuss the nature of representation theory of Lie groups, which is the chief tool to codify symmetries in field theory.

There is some information in my math primer [1], but the relevant information will be self contained in these lectures. Feel free to use both. In addition, there are the texts by Georgi [2] and Cahn [3], the latter is available for free online.

1.1 The Basics

We will begin by displaying a bunch of useful definitions and theorems about group theory in general.

Definition 1 *A* **group** *G is a set and an operation (usually called "multiplication") such that four axioms hold:*

CLOSURE: *If $a, b \in G$, then $ab \in G$.*

ASSOCIATIVE: $\forall a, b, c \in G$, $(ab)c = a(bc)$.

IDENTITY: *There exists a unique element* $e \in G$ *such that* $\forall a \in G$, $ae = ea = a$.

INVERSES: $\forall a \in G$, *there exists a unique element, denoted* a^{-1}, *such that*

$aa^{-1} = a^{-1}a = e$.

COMMUTIVITY: *If* $\forall a, b \in G$: $ab = ba$, *then the group is called an* **abelian** *group; otherwise it is called* **nonabelian**.

Definition 2 $H \subset G$ *is a* **subgroup** *of* G *if it is a subset that satisfies the above axioms of a group. Actually, it need only satisfy the closure and inverses axioms – the others follow automatically.*

Definition 3 *If* G, G' *are groups and* $\phi : G \to G'$ *is a map that preserves multiplication:* $\phi(ab) = \phi(a)\phi(b)$ $\forall a, b \in G$, *then* ϕ *is called a* **homomorphism***:*

- *If* ϕ *is one to one, it is a* **monomorphism**.
- *If* ϕ *is onto, it is an* **epimorphism**.
- *If* ϕ *is bijective, it is an* **isomorphism**.
- *If* ϕ *is an isomorphism of* G *into itself, it is an* **automorphism**.

Notice that isomorphisms are equivalence relations. We will denote isomorphic groups as $G \cong H$.

Definition 4 *If* $\phi : G \to G'$ *is a group homomorphism, then the set*

$$\text{Ker } \phi \equiv \{g \in G | \phi(g) = e'\} \ , \tag{1.1}$$

is called the **kernal** *of* ϕ. *It is easy to prove that it is a subgroup of* G *for any group homomorphism.*

Theorem 1 ϕ *is an isomorphism if and only if* Ker ϕ *is trivial. That is: it is the group of a single element (the identity).*

Definition 5 *Let $H \subset G$ be a subgroup. The set*

$$G/H \equiv \{gh|g \in G, h \in H\} \ , \tag{1.2}$$

called "G mod H" is the set of **left cosets of** H **in** G. *An element of this set is*

$$gH = [g] = \{gh|h \in H\} \ . \tag{1.3}$$

One can also define **right cosets** *similarly (denoted Hg).*

Definition 6 *$H \subset G$ subset is called a* **normal** *or* **invariant subgroup** *if and only if $gH = Hg \ \forall g \in G$. Equivalently:*

$$\forall g \in G, h \in H: \ ghg^{-1} \in H \ . \tag{1.4}$$

Normal subgroups are denoted $H \triangleleft G$.

Notice that ghg^{-1} need not equal h – as long as the product remains in H, that is enough.

Theorem 2 *All abelian subgroups are normal subgroups. Proof is trivial!*

Normal subgroups are very important in group theory and in physics. Here are three vitally important theorems about them that explain why this is so.

Theorem 3 *If $H \triangleleft G$ then G/H is a subgroup of G, called a* **factor group**. *The converse is also true.*

Theorem 4 *If $H \triangleleft G$, then there exists a homomorphism ϕ such that $H \equiv \mathrm{Ker}\ \phi$. Thus, one can think of "G/H" as "collapsing" the subgroup H to the identity.*

Theorem 5 *If $H \triangleleft G$, then $G \cong (G/H) \times H$. Similarly, if $G \cong G_1 \times G_2$, then $G_1, G_2 \triangleleft G$. Thus if $H \triangleleft G$, then so is $G/H \triangleleft G$.*

Definition 7 *If G has no normal subgroups, it is called a* **simple** *group. Notice that this does not mean that it has no general subgroups.*

While discrete group theory is an awesomely beautiful subject in its own right, and dramatically useful in so many areas of physics, we just do not have the time here to talk about them. Instead, we will be more interested in the theory of "continuous groups." These are, roughly speaking, groups with elements that differ continuously from one another. That is: all group elements can be written as $g(\vec{\theta})$ for some set of quantities $\vec{\theta}$, where g is a *continuous* function in these parameters.

Definition 8 *Let $G = \{g(\vec{\theta})\}$ be a continuous group.*

1. *If the vector of parameters $\vec{\theta}$ is a finite n dimensional vector, then G is called a* **finite group of dimension** n. *We will sometimes denote n as dim(G).*

2. *If each θ_i only assumes values over a compact space (e.g.: a closed and bounded interval) then G is called a* **compact group**.

From the above definition of a continuous group, it follows that if $\vec{\theta}_1, \vec{\theta}_2$ are continuous parameters that describe the group elements $g(\vec{\theta}_1), g(\vec{\theta}_2)$, then by the closure axiom:

$$g(\vec{\theta}_1) \cdot g(\vec{\theta}_2) = g[F(\vec{\theta}_1, \vec{\theta}_2)],$$

where F is itself continuous.

Definition 9 *If $F(\vec{\theta}_1, \vec{\theta}_2)$ described above is analytic*[1] *as well as continuous, then the group is called a* **Lie Group**.

Because of this additional condition, we can consider "small" parameters are rely on the first few terms of the expansion:

$$\boxed{g(\vec{\theta}) = 1 + iT^a\theta^a + \mathcal{O}(\theta^2)}$$

where we have normalized $g(0) = e$ (the identity, represented by the number 1) and iT^a are the linear coefficients of the expansion, called the **group generators** for reasons that will become apparent shortly.

[1]Recall that in real analysis: a function is analytic at x if it has a converging Taylor expansion at x.

Aside: What we actually have here is that a Lie group is a group that is also a *differentiable manifold*. It would take us too far afield to discuss all the rich and fascinating implications of this statement, but one very useful consequence of this is that there exists a very useful and special choice of coordinates on the Lie group manifold called **normal coordinates**, defined by an object called the **exponential map**. Consequently, we can parametrize all of the group elements as

$$g(\vec{\theta}) = e^{i\theta^a T^a}.$$

In this notation, it is also clear that $g(\vec{\theta})^{-1} = e^{-i\theta^a T^a}$. The proof of the existence of this parametrization making sense is a standard exercise in the theory of differential manifolds. This is not the place for that, but I strongly encourage you to look it up on your own.

It turns out that the structure of the group can be completely determined by working out the structure of the linear coefficients T^a, up to some topological questions that are not important at this stage. To see how, consider the group commutator:

$$[g(\vec{\theta}_1), g(\vec{\theta}_2)] \equiv g(\vec{\theta}_1) \cdot g(\vec{\theta}_2) \cdot g(\vec{\theta}_1)^{-1} \cdot g(\vec{\theta}_2)^{-1} = g(\vec{\theta}_3) \ , \tag{1.5}$$

where the last equality follows from closure of the group. Expanding the group elements for small $\vec{\theta}_i$, we get:

$$g(\vec{\theta}_3) = 1 + iT^a\theta_3^a + \mathcal{O}(\theta_3^2) = 1 + \theta_1^a\theta_2^b[T^a, T^b] + \cdots \tag{1.6}$$

where we have kept the lowest nontrivial term on each side and introduced the notation everyone knows: $[T^a, T^b] \equiv T^aT^b - T^bT^a$. This means that we need $\theta_3 \sim \theta_1\theta_2$ to be consistent, and therefore $\theta_3^2 \sim \theta^4$, so it is safe to truncate the series as we did. Comparing both sides we find:

$$\boxed{[T^a, T^b] = if^{abc}T^c} \tag{1.7}$$

for some $\mathcal{O}(1)$ coefficients f^{abc}. These are called the **structure constants** and they uniquely determine the group, at least locally.

What we have actually found is that the Lie group is determined by a **Lie Algebra**:

Definition 10 *An* **algebra** *is a vector space with a product.*

For the Lie Algebra case, this product is given by the commutator $[T^a, T^b]$, where the T^a form a basis for the algebra.

You all know examples of commutative algebras: $\mathbb{R}$ and $\mathbb{C}$. The example of a noncommutative algebra everyone knows is $\mathbb{R}^3$ with the product of two vectors being the ordinary cross product. This is an example of a Lie algbra, in fact, with $f^{abc} = \epsilon^{abc}$; this algebra is called A_1 for the experts.

> **Aside:** It is an amusing fact that there are only so many algebras you can actually write down up to isomorphism. For a simple example, consider the finite dimensional division algebras[a] $\mathcal{A}$ that preserve norms:
>
> $$|x \cdot y| = |x||y| \quad \forall x, y \in \mathcal{A}.$$
>
> It turns out that there are only four such algebras: $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$. The first two I already mentioned, and are the only *abelian* algebras with this property. The third is called the *quaternions* and the last are the *octonians*, which are not even associative.
> It is not a coincidence that there are only a finite number of "families" of Lie algebras as well, and that they are all related to the four algebras listed here, as we will see in the next aside.
>
> [a] A division algebra is one for which there is an inverse to multiplication.

The generators T^a can always be represented by matrices, and we can use this to define an inner product on the Lie algebra by using the Trace operation. There is a certain arbitrariness to this. Let us define:

$$\mathrm{Tr}\{T^a T^b\} = T\delta^{ab}. \tag{1.8}$$

T can be any number, and it also depends on how we chose to represent our generators with matrices. However, once we chose it for one particular representation, its value is fixed for all representations, as we will see below. This number is called the **Dynkin index**.

In general (as should be obvious by now) the Lie algebra is not commutative, otherwise it would be a pretty boring subject! However, there are some structure constants that do vanish, and this leads us to the following important definition:

Definition 11 *Let* $C_G \equiv \{T^a | [T^a, T^b] = 0\}$ *– that is, this is the set of generators that mutually commute with each other. These are called the* **Cartan generators** *of the algebra. It is not hard to show that they generate a subalgebra, called the Cartan subalgebra. The dimension of this subalgebra is called the* **rank** *of* G*.*

Aside: As alluded to in the previous aside, there are only a finite number of "families" of Lie algebras of a particular type. In particular: for any given rank n, there are only four (or sometimes five) simple, compact Lie algebras. They fall into four families: A_n, B_n, C_n, D_n, as well as five "exceptional" algebras:

- A_n generate $SU(n+1)$.
- B_n generate $SO(2n+1)$.
- C_n generate $Sp(2n)$ [sometimes just called $Sp(n)$ – be careful to know which convention is being used.]
- D_n generate $SO(2n)$ for $n \geq 2$.
- G_2, F_4, E_6, E_7 and E_8 are the "exceptional" Lie algebras.

In addition, there are isometries between some of the algebras and the groups they generate:

$$\begin{aligned}
A_1 = B_1 = C_1 &\longrightarrow SU(2) \cong SO(3) \cong Sp(2) \\
D_2 = A_1 \times A_1 &\longrightarrow SO(4) \cong SU(2) \times SU(2) \\
B_2 = C_2 &\longrightarrow SO(5) \cong Sp(4) \\
A_3 = D_3 &\longrightarrow SU(4) \cong SO(6)
\end{aligned}$$

The group congruences are up to global issues. All other simple, compact Lie algebras/groups are unique.
It is not a coincidence that there are four families of Lie algebras plus exceptionals, and that there are four norm preserving division algebras! In fact, $SU(N)$ describes linear transformations that leave the length of an N vector in $\mathbb{C}$ invariant; $SO(N)$ describes linear transformations that leave the length of an N vector in $\mathbb{R}$ invariant; $Sp(N)$ describes linear transformation that leave the length of an N vector in $\mathbb{H}$ invariant; and the exceptionals describe linear transformations that act on vectors in $\mathbb{O}$ – the fact that they're so strange is related to the fact that $\mathbb{O}$ is nonassociative.

1.2 Representation Theory of Lie Groups

Group theory is all well and good, a cute exercise in abstract mathematics. The million dollar question is: how do we use this information to organize a physical calculation? Tackling this question will be the subject of the rest of this lecture, and in some sense, possibly the rest of your research career!

Here we generally denote $\Gamma_N \equiv \{D(g)|g \in G\}$; $D(g)$ is an $N \times N$ matrix.

Definition 12 *A homomorphic map of a group G to the group of $N \times N$ matrices with matrix* *multiplication* Γ_N *is called a* **matrix representation** *of* G*.* N *is called the* **dimension of the representation**.

- *If this representation map is monomorphic,* Γ_N *is called a* **faithful representation**.
- *If* $D(g)$ *is a unitary matrix* $\forall g \in G$, Γ_N *is called a* **unitary representation**.

Examples:

1. $D(g) = 1 \quad \forall g \in G$ is the **trivial representation**. It is not faithful, but is still very important.
2. $[T^a]^{bc} \equiv -if^{abc}$ is called the **adjoint** or **regular representation**. It is a faithful representation. It is also clear that $\dim(\text{Adj}) = \dim(G)$.
3. Consider the Lie group (algebra) $SU(N)$ and construct the N dimensional representation. It is a theorem that this is the smallest faithful representation you can construct. It is called the **fundamental defining representation**.

The defining representation is not unique. In fact:

Theorem 6 *For a group of rank* n*, there are* n *defining representations.*

Definition 13 *Let* Γ_N *be the* N *dimensional representation of* G*. Then the set*

$$\overline{\Gamma}_N \equiv \{[D(g)]^* | g \in G,\ D(g) \in \Gamma_N\} \ ,$$

is called the **conjugate representation of** Γ_N*. If* $\Gamma_N = \overline{\Gamma}_N$*, then* Γ_N *is called a* **real representation**.

There is an ambiguity about which of $\Gamma_N, \overline{\Gamma}_N$ should be called the "conjugate" representation, since clearly $\overline{\overline{\Gamma}}_N = \Gamma_N$. We will fix this ambiguity shortly.

Now we are going to make a vital definition.

Definition 14 *A* **group action** *on a set* S *is a map* $G \times S \to S$*, generally written* $\hat{g}s$ *with* $g \in G,\ s \in S$ *such that the following axioms hold:*

IDENTITY: $\hat{e}s = s \quad \forall s \in S$;

ASSOCIATIVE: $(\hat{g}_1\hat{g}_2)s = \hat{g}_1(\hat{g}_2 s)$.

The power of all this machinery of Lie algebras is that we can use it to define a Lie group action on a vector space whose dimension is the dimension of the representation. So if Γ_N is the N dimensional representation of G, then an N vector $x^a \quad (a = 1, \dots n)$ can transform as

$$x^a \rightarrow [D(g)]^a_{.b} x^b.$$

where Einstein notation is assumed. Note the placement of indices. Similarly, we have under the conjugate representation $\overline{\Gamma}_N$

$$y_a \rightarrow [D(g)^*]_a^{;b} y_b.$$

Keeping track of covariant and contravariant indices helps keep track of conjugate representations and prevents you from making mistakes, just like in GR.

Definition 15 *A representation is* **reducible** *if every matrix can be written as a block-diagonal matrix*[2]. *We can write* $\Gamma_N = \Gamma_{N_1} \oplus \Gamma_{N_1} \oplus \cdots \oplus \Gamma_{N_p}$, *where* $\sum_{k=1}^p N_k = N$. *If a representation cannot be so decomposed, it is called* **irreducible**.

It is clear that if you know of all of the irreducible representations of a group, then you are in good shape, since any reducible representation can always be decomposed into a Kroneker sum of irreducible representations.

We can also consider the product of representations:

$$\Gamma_{N_1} \otimes \Gamma_{N_2} \equiv \{[D_1(g)]_{ab}[D_2(g)]_{cd} | \forall g \in G\}.$$

Such objects are generally reducible, and we would like to understand how to decompose products like this into irreducible representations:

$$\Gamma_{N_1} \otimes \Gamma_{N_2} = \Gamma_{N_1} \oplus \Gamma_{N_1} \oplus \cdots \oplus \Gamma_{N_p} \qquad \text{where} \quad \sum_{k=1}^{p} N_k = N_1 \cdot N_2.$$

[2]Technically, this is a stronger condition called **completely reducible**; but for everything we will be doing, all reducible representations are completely reducible, so I will not distinguish in the future.

Theorem 7 *For a vector in an (irreducible) representation, the elements of the vector are described by the n eigenvalues of the Cartan generators.*

This is just a generalization of the spin algebra from quantum mechanics. $SU(2)$ ($= A_1$) is a rank 1 algebra, and inside a given representation, the elements are indexed by the eigenvalues of the one Cartan generator (J_3). These eigenvalues are called **weights**. This suggests a very powerful tool for constructing representations: Let $\{\vec{e}_1, \cdots, \vec{e}_n\}$ be a set of orthonormal vectors ($n =$rank(G)) that transform in the different defining representations:

- $\vec{e}_1$ is the fundamental representation (Γ_1).
- $\vec{e}_{n-i+1}$ transforms in the conjugate representation of $\vec{e}_i$.

This set of vectors provide a basis for any quantity that transforms under any representation of the Lie group, generally called a **tensor operator**:

$$\hat{\mathcal{O}} = O_{a_1 \cdots a_i; b_1 \cdots b_j; \cdots}(e_1^{a_1} \cdots e_1^{a_i})(e_2^{b_1} \cdots e_2^{b_j}) \cdots \qquad (1.9)$$

where all indices of a given letter are symmetrized and all conjugate indices are trace free. All tensor operators are products of copies of the defining representations:

$$g : O_{a_1 \cdots; b_1 \cdots; \cdots} \to [D_1(g)]_{a_1 \bar{a}_1} \cdots [D_2(g)]_{b_1 \bar{b}_1} \cdots O_{\bar{a}_1 \cdots; \bar{b}_1 \cdots; \cdots} \qquad (1.10)$$

In particular, an operator that transforms under an irreducible representation of G is described completely as an operator with i indices of type 1, j indices of type 2, etc.

In summary: **Irreducible representations of a Lie group of rank n are described by n positive integers.**

Examples:

1. $SU(2)$ – This is a rank 1 group, so irreducible representations are defined by a single integer:
   - $(1): \quad \xi^a$
   - $(4): \quad \xi^{abcd}$
2. $SU(3)$ – Rank 2, so it is described by two integers:

- $(0,0)$: Trivial rep
- $(1,0)$: Fundamental (ξ^a)
- $(0,1)$: Antifundamental ($\overline{\xi}_a$)
- $(1,1)$: Adjoint (ξ^a_b)

3. $SU(4)$ – Rank 3, needing three integers:

- $(0,0,0)$: Trivial
- $(1,0,0)$: Fundamental (ξ^a)
- $(0,0,1)$: Antifundamental ($\overline{\xi}_a$)
- $(1,0,1)$: Adjoint (ξ^a_b)
- $(0,1,0)$: A new defining representation with one index[3] ($\xi^{\hat{a}}$).

From these examples, we can see a few patterns:

1. $(0,\cdots,0)$ is always the trivial representation.
2. $(1,0,\cdots,0)$ is the fundamental representation.
3. $(1,0,\cdots,0,1)$ is the adjoint representation.
4. $\overline{(a,b,\cdots,c)} = (c,\cdots,b,a)$.

From this last point, we see that the adjoints are always real representations. So is $(0,1,0)$ in $SU(4)$.

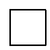
Unfortunately, $(a,b,\cdots,c)\otimes(a',b',\cdots,c')$ and $(a,b,\cdots,c)\oplus(a',b',\cdots,c')$ are not easily visualized in this notation. Fortunately, there is a very powerful and useful notation for keeping track of representations, and also for computing representations in products. This is the technology of Young tableaux.

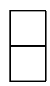
[3]There is no single accepted notation for these kinds of indices that I am aware of, so I put a hat over it. As we will see below, it is a real represntation so it does not matter if you put it upper or lower, but in general it would.

1.3 Young Tableaux for $SU(N)$

Every group has a pictorial mnemonic for keeping track of representations through a series of boxes lined up in a given way. These collections of boxes are called **Young Tableaux**, named after an accountant who developed them as a means of keeping track of permutations (they were originally designed for the discrete group S_n). We, however, will exclusively use them for describing representations of the $SU(N)$ groups. But they can also be used for any other group with only slight modification of the rules.

Consider $SU(n+1)$, which recall has rank n. An irreducible representation of this group is described by n integers.

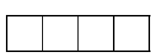
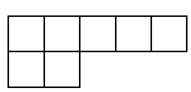
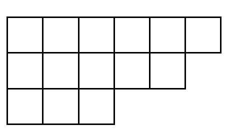
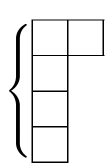
For each $(1, 0, \cdots, 0)$: assign a  .

For each $(0, 1, \cdots, 0)$: assign a  .

$\vdots$ $\vdots$

where the m^{th} integer is represented by a column of m boxes. We already pointed out that any representation can be constructed by a symmetrized product of defining representations. This is realized through Young tableaux by juxtaposing columns into a single large tableau, with the convention that *all Young tableaux are top-left justified.*

Examples:

1. (4) in $SU(2)$: 
2. $(3, 2)$ in $SU(3)$: 
3. $(1, 2, 3)$ in $SU(4)$: 
4. Adjoint $[(1, 0, \cdots, 0, 1)]$ in $SU(N)$: $N-1$ 

1.3.1 Computing dimensions of $SU(N)$ reps with Young tableaux

Young tableaux are wonderous devices – not only are they a powerful mnemonic for keeping track of representations, but they allow you to compute the dimensions of the representations almost without effort! Again, we will be considering $SU(N)$, but the rules can be altered easily enough to cover any group. But for the sake of this lecture, let us consider an arbitrary Young tableau describing a representation of $SU(N)$. To compute the dimension, there are two steps:

1. In the top left box, put an N. Going down the column, label the boxes counting down. Going across the rows left to right, count up. Take the product of all of these numbers. Call it F.

2. Starting on the right side of the tableau, draw all the "hooks" that you can – these are lines that go right to left, and make a single, hard turn south and continue down. Count the number of boxes each hook passes through, and take the product of all of these numbers. Call it H.

Theorem 8 $\boxed{\dim(\text{Y.T.}) = \frac{F}{H}}$

Examples:

1. (4) in $SU(2)$: $\begin{array}{|c|c|c|c|}\hline 2 & 3 & 4 & 5 \\ \hline\end{array}$ $\dim = \frac{2\cdot3\cdot4\cdot5}{1\cdot2\cdot3\cdot4} = 5.$

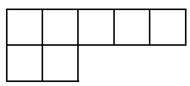
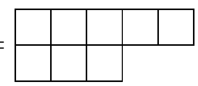
2. $(3,2)$ in $SU(3)$: $\begin{array}{|c|c|c|c|c|}\hline 3 & 4 & 5 & 6 & 7 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$ $\dim = \frac{3\cdot4\cdot5\cdot6\cdot7\cdot2\cdot3}{1\cdot2\cdot3\cdot5\cdot6\cdot1\cdot2} = 42.$

3. $(0,1,0)$ in $SU(4)$: $\begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline\end{array}$ $\dim = \frac{4\cdot3}{2\cdot1} = 6.$

4. Adjoint $[(1,0,\cdots,0,1)]$ in $SU(N)$: $N-1\left\{\begin{array}{|c|c|}\hline N & \\ \hline \vdots \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\right.$ $\dim = \frac{(N+1)!}{N\cdot(N-2)!} = N^2-1.$

A word on notation: often a representation is described by its dimension. For example, in $SU(3)$, the $(1,0)$ is sometimes called the $\mathbf{3}$ while the $(0,1)$ is called the $\bar{\mathbf{3}}$; in $SU(4)$, the $(0,1,0)$, which is a real representation, is called the $\mathbf{6} = \bar{\mathbf{6}}$. This is a very common notation, but unfortunately it can be a little confusing, since there could be more than one representation with the same dimension. So you must be careful to know what convention the author is using when you see such a labeling of dimensions. The tradition is to use the conventions of [4].

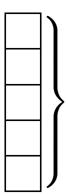
From the above examples, it is also clear that to turn a representation to the conjugate representation, all you need to do replace each column of x boxes with a column of $N - x$ boxes, since that corresponds to reflecting the integers in the weight vector. To decide whether a rep should be barred or not, we use the rule: *If the largest integer is to the left (least number of long columns in the Young tableau), the representation is not barred.* So for example, in $SU(3)$:

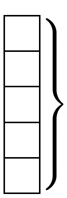
$\mathbf{42} = (3, 2) =$  $\Leftrightarrow$ $\overline{\mathbf{42}} = (2, 3) =$ 

1.3.2 Decomposing Products of Irreducible Representations of $SU(N)$ with Young Tableaux

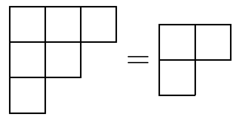
We are finally in a position to understand how to decompose products of representations in terms of a direct sum of irreducible representations. Again, Young tableaux make the whole process deliciously simple!

As a first step, take a moment to notice a few facts about columns in a typical Young tableau in $SU(N)$:

1. dim  N boxes $= 1$, so this is a singlet.

2.  $N + 1$ boxes does not exist.

This means that when we construct Young tableaux and end up with N boxes in a column, we can drop the whole column (since it is just 1) and we *never* can construct a Young tableau wiht more than N boxes in a column. For example, in $SU(3)$:



It is sufficient to consider the product of two irreducible representations, since more general products can be built up by associativity of the Kroneker product. Also note that this product is commutative, so we can arrange the Young tableaux in any order that we wish. This will be useful in general. Now, to proceed:

To compute $\Gamma_1 \otimes \Gamma_2$:

1. Chose the Young tableau with the least number of boxes (without loss of generality since the Kroneker product is commutative, let it be Γ_2) and place an "a" in every box in the first row, a "b" in every box in the second row, etc.

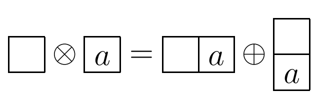
2. Start adjoining "a" boxes onto Γ_1's tableau to the **right and below**, being careful to maintain the justification rules, and with **no more than one "a" in each column**. This last rule is there to avoid both symmetrizing and antisymmetrizing in the same index (which would give zero).

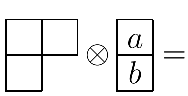
3. Now do the same thing with the "b" boxes with one additional rule: **the number of as above and to the right of the first b must be greater than or equal to the total number of bs**. This rule is there to avoid double counting.

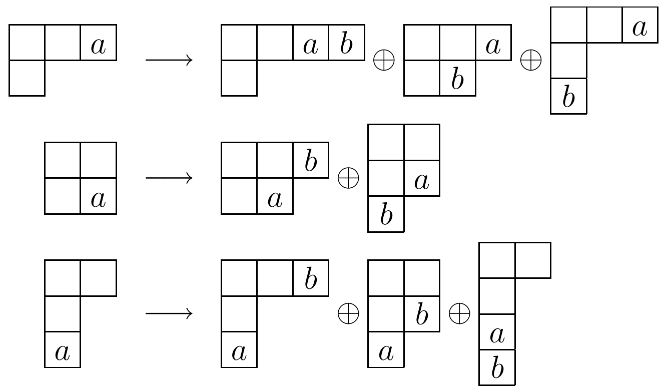
4. Repeat wtih $c, d, \ldots$ with Step 3 being generalized accordingly.

5. Now go through the list of tableaux, killing off any vanishing diagrams or singlet columns as mentioned above. What is left is your answer!

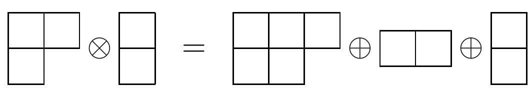
Examples: $SU(2)$ is a little too trivial, so let us consider $SU(3)$:


$\Rightarrow \mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \overline{\mathbf{3}}$



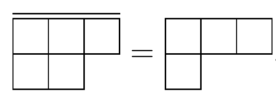


The first diagram in each row violates the rule about as before bs, as does the second diagram in the third row. The last diagram in this chain has four boxes in a column, and since this is $SU(3)$ this diagram vanishes. Finally, the last diagram in the first two rows have singlet columns that we can cross out. So our final result is:



$$(1,1)\otimes(0,1) \;=\; (1,2)\oplus(2,0)\oplus(0,1)$$

$$\mathbf{8}\otimes\bar{\mathbf{3}} \;=\; \overline{\mathbf{15}}\oplus\mathbf{6}\oplus\bar{\mathbf{3}}$$

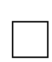
where I have included three ways to denote the results. Notice: , where $\dim = \frac{3\cdot4\cdot5\cdot2}{4\cdot2} = 15$. But the $\mathbf{15}$ we have has more long columns – therefore it is the conjugate representation.

1.4 Group Theory Coefficients

There are some very general relationships among products of generators of Lie groups that appear when performing calculations. Through the orthonormality condition, we can write very general rules for what these products are. In particular, there are three quantities of practical interest:

Dynkin Index: We've seen this one already:

$$\boxed{\mathrm{Tr}[T_R^a T_R^b] = T(R)\delta^{ab}} \tag{1.11}$$

If we specify $T(F)$ in the fundamental representation , then the Dynkin index is unique and a function of the representation. It is standard to use $T(F) = \frac{1}{2}$, although

$T(F) = 1$ is also sometimes used. **Be careful** to know what the convention is, or you will make terrible mistakes!

Quadratic Casimir: This is analogous to the J^2 operator in quantum mechanics:

$$\boxed{T_R^a T_R^a = C_2(R)\mathbf{1}_R} \tag{1.12}$$

where there is the usual sum over a and $\mathbf{1}_R$ is the $\mathrm{dim}R \times \mathrm{dim}R$ unit matrix. Fixing κ_F above means that $C_2(R)$ is also unique and a function of R.

Anomaly Coefficient: The totally symmetric product of three generators appears often, as we will see. From closure of the algebra, we can define the totally symmetic three-index symbol in the fundamental representation of $SU(N)$:

$$[T_F^b, T_F^c]_+ = \frac{1}{N}\delta^{ab} + d^{abc}T_F^c \ . \tag{1.13}$$

These objects are not fundamental: they can be computed in terms of the structure constants – but they are a useful shortcut. Using this tool we can compute for the totally symmetric product of three generators in any representation:

$$\boxed{\mathrm{Tr}\{T_R^a[T_R^b, T_R^c]_+\} = A(R)T(F)d^{abc}} \tag{1.14}$$

The $A(R)$ are called anomaly coefficients and can be computed or looked up in a table. Notice that they are normalized so that $A(F) = 1$. Also notice that the existence of a nontrivial d^{abc} is only true for $SU(N)$ groups for $N \geq 3$, so that these can only exist for such groups. In particular, $Sp(N)$ and $SO(N)$ (except $SO(6)$) do not have nonvanishing anomaly coefficients, nor does $SU(2)$. This is a very important fact that will come up again later.

The Dynkin index, quadratic Casimir and anomaly coefficient can all be looked up in tables for any irreducible representation of $SU(N)$, and of the other groups as well. We also have a very useful formula relating the Casimir and Dynkin index:

$$C_2(R) = \left(\frac{\mathrm{dim}G}{\mathrm{dim}R}\right)T(R) \ . \tag{1.15}$$

It is also clear from the rules of the Trace operation that:

1. $T(R_1 \oplus R_2) = T(R_1) + T(R_2)$,

2. $T(R_1 \otimes R_2) = (\text{dim}R_1)T(R_2) + (\text{dim}R_2)T(R_1)$.

The anomaly coefficients have similar rules, with the extra useful rule that:

$$A(\overline{R}) = -A(R) \ . \tag{1.16}$$

In particular, this means that an object that transforms in a real representation (either an irreducible real representation like the Adjoint representation or a vector-like representation $R \oplus \overline{R}$) automatically has a vanishing anomaly coefficient. This will be very important in the future. This is also another way to see that $SO(N)$ and $Sp(N)$ algebras cannot have nonvanishing anomaly coefficients, since they only admit real representations.

Chapter 2

Speaking the Language: How to Build Models

All throughout the first lecture, we have implicitly been assuming that our symmetry group was *linearly realized* – that is, our group elements took the form of linear transformations (matrices with constant entries) acting on some vectors. However, there is no physical reason to assume this, and in fact, as we will see in this lecture, there are plenty of physical reasons to abandon it! To this end, we would now like to take the next step and analyze *nonlinear realizations* of a symmetry group.

The reason why nonlinear realizations of a symmetry group are so useful is that it turns out to be the most effective way to represent a symmetry that has been *spontaneously broken.* This, as we will see, is a most unfortunate name, as a "spontaneously broken symmetry" is not actually broken! This leads many students to confusion, and so I would like to take some time to study how spontaneously broken symmetries can be realized nonlinearly in an action.

The technology to do this was first worked out in [5, 6]. These papers, published back to back in the Physical Review, are required study material for any student serious about learning quantum field theory, but be warned: they are particularly difficult papers to get through. I recommend you read them very carefully and think hard about what they're trying to say. I will summarize the ideas they present, avoiding proofs where I think I can get away with it. But you should go and work these things out yourselves so that you are sure to understand them. This technology is, in many ways, the core to all effective field theories.

I am not going to spend any time discussing the ideas of spontaneous symmetry breaking and its relationship to Noether's Theorem. I am assuming that this is already known. For a review, see your favorite textbook, such as [7].

2.1 Nonlinear Realizations of Symmetries

Consider a group G, and a subgroup $H \subset G$. A **nonlinear realization** on G is a representation of the direct product $H \otimes (G/H)$. If $H \triangleleft G$, then G/H is a group and this statement is nearly vacuous. However, if H is *not* normal, as is the case whenever G is a simple group for example, then G/H is *not* a group! It is merely the set of left cosets of H (see Section 1.1).

In particular: If V^a are the generators of H and A^a are the remaining generators of G, then we can chose a representation of the group to take the form:

$$\boxed{g(\xi, u) = e^{i\xi\cdot A}e^{iu\cdot V}} \tag{2.1}$$

where $e^{iu\cdot V} \in H$ and $e^{i\xi\cdot A} \in G/H$. Notice again that in general, the set of all $e^{i\xi\cdot A}$ does not form a group. For a general $g \in G$, we have from closure of G that

$$ge^{i\xi\cdot A} = e^{i\xi'\cdot A}e^{iu'\cdot V} , \tag{2.2}$$

where $\xi' = \xi'(\xi, g)$ and $u' = u'(\xi, g)$ are analytic functions due the Lie group structure.

Now back to physics. Suppose we have a theory containing several fields ψ with an action that is invariant under the whole group G. Let us chose to realize this symmetry by having all these fields in a linear representation of H:

$$h \ : \ \psi \longrightarrow \psi' \equiv D(h)\psi . \tag{2.3}$$

Then observe the action of G on the object $\hat{\psi} \equiv e^{i\xi\cdot A}\psi$, using associativity:

$$g(e^{i\xi\cdot A}\psi) = e^{i\xi'\cdot A}e^{iu'\cdot V}\psi = e^{i\xi'\cdot A}\psi' = \hat{\psi}' . \tag{2.4}$$

Thus we have a new, *nonlinear* realization:

$$\boxed{g \ : \ \xi \longrightarrow \xi', \qquad \psi \longrightarrow D(e^{iu'\cdot V})\psi} \tag{2.5}$$

Notice what has happened. $\xi \to \xi'$ is a well-defined transformation independent of the fields in the action; but in order for these fields ψ to have well-defined transformation laws, we needed ξ, since u' is a function of ξ. That is why these are called nonlinear realizations. The ξ^a are new objects that define the group action of the fields of our theory. When the symmetry is spontaneously broken $G \to H$, these fields develop dynamics of their own, and they represent the **Goldstone Modes** of the model. They have appeared in a totally different way than what is usually shown – they were required in order for us to be able to define a nonlinear realization.

There is more than one way to parametrize these modes. A very useful way is to use the exponential map to write $\Sigma \equiv e^{i\xi\cdot A}$. When written in this form, the dynamics of the Goldstone modes is given by the infamous "nonlinear sigma model," which we shall discuss next.

2.2 The Nonlinear Sigma Model

Consider N complex scalar fields ϕ^a in a generic action

$$S = \int d^4x \left\{ |\partial_\mu \phi|^2 - V(\phi) \right\} \ . \tag{2.6}$$

Ignoring $V(\phi)$ for the moment, we have a built in $SU(N)$ symmetry:

$$\phi \longrightarrow U\phi \qquad U^\dagger U = \mathbf{1} \ . \tag{2.7}$$

Let us realize this symmetry nonlinearly by defining

$$\hat{\phi} = e^{i\xi'\cdot A}\phi \equiv \Sigma\phi \ , \tag{2.8}$$

where the A^a are a set of generators whose complement generates a subgroup[1] H of $SU(N)$. Then the kinetic term in the Lagrangian can be written in terms of this new field:

$$\begin{aligned} \mathcal{L}_0 &= \partial_\mu(\hat{\phi}^*_a \Sigma^a_b)\partial^\mu(\Sigma^{\dagger b}_c \hat{\phi}^c) \\ &= |\hat{\phi}|^2 \mathrm{Tr}[\partial_\mu \Sigma^\dagger \partial^\mu \Sigma] + |\partial_\mu \hat{\phi}|^2 + \mathrm{Tr}[j^\mu A_\mu] \ , \end{aligned} \tag{2.9}$$

[1]For example, $H = SU(N-1),\ SO(N)$, or the trivial group.

where

$$[j^\mu]^a_b = i\hat{\phi}^*_b \overleftrightarrow{\partial^\mu} \hat{\phi}^a , \tag{2.10}$$

$$A_\mu = \Sigma^\dagger \partial_\mu \Sigma , \tag{2.11}$$

and I have used $\Sigma^\dagger\Sigma = \mathbf{1}$. Funny, does this A_μ remind you of something??

Now if we make the assumption that $V(\phi) = V(\phi^*\phi)$, a very reasonable assumption in practice, then the change of field (2.8) does not affect the potential. Dropping the hat, we therefore have:

$$S = \int d^4x \left\{ |\phi|^2 \mathrm{Tr}[\partial_\mu \Sigma^\dagger \partial^\mu \Sigma] + |\partial_\mu \phi|^2 + \mathrm{Tr}[j^\mu A_\mu] - V(\phi) \right\} . \tag{2.12}$$

Why did we go through all this nonsense? It looks worse than before! But now imagine that $V(\phi)$ picks out a vev for $|\phi| = v$ through the mechanism of spontaneous symmetry breaking[2] $G \to H$. Now the derivative terms vanish and we can ignore the current j^μ as well as the kinetic term for ϕ. The potential gives a constant at the minimum which we can also ignore, and our final result is

$$\boxed{\mathcal{L} = v^2 \mathrm{Tr}[\partial_\mu \Sigma^\dagger \partial^\mu \Sigma]} \tag{2.13}$$

This is the action of the **nonlinear sigma model** and it comes out quite generically in nonlinear realizations of spontaneously broken symmetries.

2.3 Effective Field Theories and the Chiral Lagrangian

Now that we understand how symmetries can be realized nonlinearly with the explicit introduction of Goldstone modes. and how when a symmetry is spontaneously broken those Goldstone modes develop dynamics in the form of the nonlinear sigma model, we would like to use this powerful new tool to develop a methodology for doing physics. Because it is so very famous, and carries with it almost all the features of effective field theories in general, we will spend some time discussing the Chiral Lagrangian. This was originally used to understand the strong nuclear force and the low energy behavior of mesons and baryons. However, it is a very generic model whose features can be applied to almost anything.

[2]For example, if $\langle \phi^a \rangle = v\delta^{a1}$, then $H = SU(N-1)$ is still a linearly realized symmetry.

I am not going to discuss the details of meson physics here – that is a topic unto itself, and not necessarily of interest to everyone in the audience. For those that are interested, there are plenty of references out there that talk in great detail about it. A very good reference, for this and so many other things, is [8].

In addition, at this time I'm going to start using two-component spinors, something that many people do not use. Since the chiral lagrangian is, well, *chiral*, it makes sense to do things in terms of Weyl spinors. Also, for any students interested in going to to study supersymmetry and the MSSM, another chiral theory, Weyl spinors play a predominant role there as well. I have provided a lightning review of what is important to understand this lecture in the appendix to this chapter; for more details take a look at the excellent review just published [9].

The general scheme for constructing an effective theory follows four basic steps:

1. Identify the degrees of freedom and physical scales in the problem.
2. Identify the symmetries of these degrees of freedom by encoding them into fields and writing down their free (kinetic) actions.
3. Write all the interactions allowed by the symmetries, *including* source terms that are also allowed to transform under the symmetries.
4. Fix the various sources based on what you want to describe.

We will spend the rest of this lecture discussing the canonical example of this algorithm.

2.3.1 Constructing the Chiral Lagrangian

Below roughly 1 GeV, it is known that there are three spin $\frac{1}{2}$ degrees of freedom (u, d, s) that are nearly massless, each behaving as a color triplet:

$$\mathcal{L}_0 = A^a_{Lb}\overline{q}_{La} i \not{D} q^b_L + A^a_{Rb}\overline{q}_{Ra} i \not{D} q^b_R \ , \qquad (2.14)$$

where $\not{D} \equiv \overline{\sigma} \cdot D$, and D_μ is the QCD covariant derivative. Recall that in four dimensions, a spin $\frac{1}{2}$ degree of freedom is normally described by *two* Weyl fields. In general, these fields have nothing to do with each other, and there is a symmetry in this action given by

$U(3)_L \times U(3)_R$, under which $q_L = (\mathbf{3}, 0)$ and $q_R = (0, \overline{\mathbf{3}})$, with barred fields in the conjugate representation[3]. We can use this *chiral* symmetry to independently set $A_L = A_R = \mathbf{1}$. Also note that in this two-component notation, *all* spinors are in the $(\frac{1}{2}, 0)$ representation, while the barred spinors are in the $(0, \frac{1}{2})$ representation, regardless of the L, R label. If you are uncomfortable with this, see the appendix.

In this basis, we can add to 2.14 some source terms:

$$\mathcal{L} = \mathcal{L}_0 + \overline{q}_L(\overline{\sigma} \cdot l)q_L + q_R(\sigma \cdot r)\overline{q}_R - q_L(s - ip)q_R - \overline{q}_R(s + ip)\overline{q}_L \ , \tag{2.15}$$

where l_μ, r_μ are vector sources in the adjoint representation of $U(3)_{L,R}$ and s, p are real scalar and pseudoscalar sources in the bifundamental representation. So in summary, (2.15) is invariant under the following transformations:

$$q_L \longrightarrow Lq_L \ , \tag{2.16}$$

$$q_R \longrightarrow q_R R^\dagger \ , \tag{2.17}$$

$$l_\mu \longrightarrow Ll_\mu L^\dagger + iL\partial_\mu L^\dagger \ , \tag{2.18}$$

$$r_\mu \longrightarrow Rl_\mu R^\dagger + iR\partial_\mu R^\dagger \ , \tag{2.19}$$

$$(s + ip) \longrightarrow L(s + ip)R^\dagger \ . \tag{2.20}$$

Notice that we are allowing the chiral transformations to be local in general; this will be important later when we talk about how to embed electromagnetism into the picture.

Now at these energies, QCD is nonperturbative, and we must look to some other expansion. Fortunately, one is available: a derivative (or energy/momentum) expansion. It is also believed that QCD dynamics spontaneously breaks the chiral symmetry to the vector subgroup:

$$SU(3)_L \times SU(3)_R \longrightarrow SU(3)_D \ ,$$

defined by[4] $L = R$. It does this by giving a vev to the chiral condensate

$$\langle q_L q_R \rangle = \sigma \qquad \sigma \longrightarrow \sigma(LR^\dagger) \ . \tag{2.21}$$

This does seem to happen from experiment and lattice calculations, although *how* this actually happens is still not strictly understood.

[3]Notice that q_R is is in the *conjugate* representation of $SU(3)_R$ – it is the *anti*quark.

[4]You'll notice that I've pulled a fast one and dropped the $U(1)$ factors! We'll come back to those later.

Since we have a spontaneously broken symmetry, we know that we should look for a nonlinear realization of the full chiral symmetry (again ignoring the $U(1)$s for now) that are linear on the vector subgroup[5]. So let us write:

$$L = UV \ , \tag{2.22}$$

$$R = U^\dagger V \ , \tag{2.23}$$

where $V \in SU(3)_D$ and $U \in SU(3)^2/SU(3)_D$. From this we see that the chiral condensate (2.21) transforms as

$$LR^\dagger = U^2 \equiv \Sigma \ , \tag{2.24}$$

where $\Sigma \equiv e^{2i\Pi/f}$ and $\Pi \equiv \Pi^a T^a$ are the Goldstone modes. Written in this form, it is clear that under a *full* $SU(3)^2$ transformation,

$$\Sigma \longrightarrow L\Sigma R^\dagger = U(V\Sigma V^\dagger)U \ . \tag{2.25}$$

So Σ is an adjoint (octet) under the $SU(3)_D$ and something rather funny under the coset space. Notice the amazing power of the nonlinear sigma model: the Π fields have horribly complicated nonlinear behavior under the transformations (as can be seen from a brute force expansion of (2.25)), but the Σ field transforms rather simply.

Now that we have worked out the degrees of freedom and the symmetries, we can proceed to step 3 in our schema and try to write down all the interactions that are allowed by the symmetries:

1. No derivatives: there are no interactions here other than a constant, since $\Sigma^\dagger\Sigma = \mathbf{1}$. However, there is the source terms involving $(s \pm ip)$

2. One derivative: there are no interactions here since there is nothing to couple the Lorentz index with except the vector sources, but this is forbidden by gauge invariance.

3. Two derivatives: Here we have the nonlinear sigma model kinetic term, (2.13). However, we must now convariantize the derivatives.

[5] In this particular case, the vector subgroup is a normal subgroup, so the cosets also form a subgroup called the **axial subgroup**. However, this is purely a coincidence and as I mentioned before, does not happen in general.

So we find that the most general action to two derivatives is:

$$\boxed{\mathcal{L} = \frac{f^2}{2}\left\{\mu\mathrm{Tr}[(s+ip)\Sigma] + \mu\mathrm{Tr}[\Sigma^\dagger(s-ip)] + \frac{1}{2}\mathrm{Tr}[D_\mu\Sigma D^\mu\Sigma]\right\} - \frac{1}{4g_L^2}L_{\mu\nu}L^{\mu\nu} - \frac{1}{4g_R^2}R_{\mu\nu}R^{\mu\nu}} \tag{2.26}$$

where f and μ are parameters with dimensions of mass, the $\frac{1}{2}$s are chosen to canonically normalize the Π fields, and

$$D_\mu\Sigma \equiv \partial_\mu\Sigma - il_\mu\Sigma + i\Sigma r_\mu \ , \tag{2.27}$$

$$L_{\mu\nu} \equiv \partial_\mu l_\nu - \partial_\nu l_\mu + [l_\mu, l_\nu] \ , \tag{2.28}$$

and similarly for $R_{\mu\nu}$. Notice that these covariant derivatives are forced on us since the vector sources can be thought to come from a local symmetry.

Notice the all important fact that there is no longer any mention of QCD in (2.26) – the only remaining part of QCD is the unknown parameters f and μ. In principle, we could calculate these parameters in QCD, but since we don't know how to use QCD consistently at these low energies, these numbers become phenomenological parameters that must be measured. However, once they are measured, then we can start to make quantitative predictions with this theory. *This* is the power of effective field theory: it allows you to compartmentalize everything you don't know, and then you can proceed to calculate physical observables in terms of these quantities.

There is plenty of things we can do at this point. We can use this general Lagrangian to incorporate other important physics beyond QCD. This is a vital next step, and we will consider two such examples: quark masses and QED.

Quark Masses

A mass term for the quarks explicitly breaks the chiral symmetry. The interaction at the quark level is

$$\Delta\mathcal{L}_M = q_L M q_R \ + \ \text{h.c.} \ . \tag{2.29}$$

But notice that *if* M was a field that transformed as $M \to LMR^\dagger$, then it would be okay. But wait! We have such a source field already – $(s+ip)$! So if we take (2.26) and set

$$\langle s\rangle = M \qquad \langle p\rangle = 0 \ , \tag{2.30}$$

then we have completely incorporated quark mass effects to this order in the derivative expansion. Wow!

QED

The quarks are electrically charged, with $Q_u = +\frac{2}{3}$ and $Q_d = Q_s = -\frac{1}{3}$. Therefore, this will also break the chiral symmetry. The quark level QED vertex is

$$\Delta\mathcal{L}_{\text{QED}} = \overline{q}_{Li}(\overline{\sigma} \cdot A) q_L^j Q_j^i \ + \ (L \to R) \ . \tag{2.31}$$

Since QED couples to *both* L and R fields equally (it is a vectorlike theory) we know that we can use:

$$A_\mu Q_j^i = \frac{1}{2}(l_\mu + r_\mu) \ . \tag{2.32}$$

Now you can work out which components of $(l_\mu^a T_L^a)_j^i$ and $(r_\mu^a T_R^a)_j^i$ have to vanish and which get vevs to facilitate this identification. I leave this to you for homework.

This trick of capturing symmetry breaking effects by identifying the sources in an effective Lagrangian is known as a **spurion analysis**. This is another one of those gems in the effective field theorist's bag of tricks!

2.3.2 Higher Order Effects and Naive Dimensional Analysis

One of the most awesome things about effective field theories is that they can tell you exactly how they fail! How many people do you know that can do that?!

Consider higher order effects that enter into a calculation. They can come from two places:

1. Loop diagrams, generally divergent, that come from multiple insertions of operators.

2. 4-derivative (and higher) operators that we ignored in (2.26).

We have the following cardinal rule of effective theories:

These contributions, both occurring at the same order in the energy/momentum expansion, should give the same order of magnitude answer!

If this rule were to be violated, then it would invalidate the expansion. As we will see, this rule carries with it some powerful corollaries.

First, let us consider the $\mathcal{O}(E^4)$ terms in the effective theory. These operators can have derivatives, mass insertions or involve photon insertions, leading to operators like

- $\mathrm{Tr}[D_\mu \Sigma D_\nu \Sigma^\dagger D^\mu \Sigma D^\nu \Sigma^\dagger]$
- $\mathrm{Tr}[(s+ip)\Sigma(s+ip)\Sigma]$
- $\mathrm{Tr}[L_{\mu\nu}\Sigma R^{\mu\nu}\Sigma^\dagger]$

It turns out, after a great deal of simplification, there are a total of ten such operators for an $SU(3)$ chiral Lagrangian.

What are the coefficients of these operators? Just by dimensional analysis, an operator of dimension d must have a coefficient that goes like

$$c\frac{f^2}{\Lambda_\chi^{d-2}}\ , \tag{2.33}$$

where c is a dimensionless number, and the f^2 factor is there to match to the normalization we used when defining (2.26). Λ_χ is the **chiral symmetry breaking scale** and represents the energy where the expansion breaks down. In other words, we are doing an expansion in E/Λ_χ.

The dimensionless coefficients, like all the other parameters, can in principle be computed in QCD, but in practice are either measured or computed on a lattice. Since there are no more large or small numbers to play with, it must be that these coefficients are $\mathcal{O}(1)$. If we were to measure them to be substantially different from unity, it would imply that we are missing some physics. This is called the **naturalness** argument. For the case of pion physics, these parameters are indeed $\mathcal{O}(1)$ numbers. In other models, such as the electroweak model of particle physics, they can potentially be quite large (or small). This is the source of the "naturalness problem" of particle physics. It tells us that the model we have is probably missing some important dynamics.

We have seen one way that effective field theories can tell us if something is wrong – that of naturalness constraints. What else can we learn from higher order terms in the expansion?

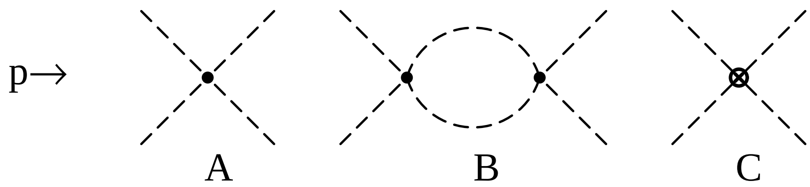


Figure 2.1: 2 → 2 Goldstone boson scattering diagrams at NLO in the chiral Lagrangian. The solid dot and circlecross vertices come from the $\mathcal{O}(E^2)$ and $\mathcal{O}(E^4)$ terms in the action, respectively. Not shown are the crossing diagrams of B.

To see what can be done, let us consider the $2-2$ scattering of the Goldstone modes coming from the diagrams in Figure 2.1. Without being too concerned about the details, we can compute:

$$\begin{aligned}
\mathcal{M}_A &\sim \frac{p^2}{f^2} \\
\mathcal{M}_B &\sim \int \frac{d^4q}{(2\pi)^4} \frac{(p+q)^2/f^2 \cdot (p-q)^2/f^2}{(q^2)^2} \sim \frac{1}{16\pi^2}\left[\Lambda^2 \frac{p^2}{f^4} + \frac{p^4}{f^4}\log\left(\frac{\Lambda}{\mu}\right)\right] \\
\mathcal{M}_C &\sim c\frac{p^4}{f^2\Lambda_\chi^2}
\end{aligned}$$

where Λ is a momentum cutoff (not to be confused with the chiral symmetry breaking scale) and μ is the subtraction point that comes along whenever you renormalize. Summing up these contributions, we find:

$$\mathcal{M}(\pi\pi \to \pi\pi) \sim \frac{p^2}{f^2}\left[1 + \frac{1}{16\pi^2}\left(\frac{\Lambda}{f}\right)^2\right] + \frac{p^4}{f^4}\left[\frac{1}{16\pi^2}\log\left(\frac{\Lambda}{\mu}\right) + \frac{cf^2}{\Lambda_\chi^2}\right] + \text{finite} \; . \qquad (2.34)$$

The first term is a pure renormalization of f, and we need not worry about it. The second term, on the other hand, is deep! Recalling that the final answer cannot depend on μ, we can ask what happens by changing $\mu \to \mu e$. It must be compensated for by an appropriate change in c:

$$\mu \to \mu e \quad \Rightarrow c \to c - \frac{\Lambda_\chi^2}{16\pi^2 f^2} \qquad (2.35)$$

But wait – we already said that c must be $\mathcal{O}(1)$ to maintain the perturbative expansion. Then we conclude that in order for this theory to make sense:

$$\boxed{\Lambda_\chi \sim 4\pi f} \qquad (2.36)$$

Wow! By insisting that the chiral Lagrangian had to make sense, the effective theory had the decency to tell us precisely how seriously we can take it! The result in (2.36) is quite generic, and is called **naive dimensional analysis**, or NDA. For the case of pion physics, $f_\pi \sim 100$ MeV, so we predict that $\Lambda_\chi \sim 1$ GeV, and we expect the chiral Lagrangian to work only for energies much below this scale. Sure enough: there is a new particle that is not included in our theory, called the ρ meson[6], with a mass at 770 MeV. Is this a coincidence? I think not!

At the scale Λ_χ, the effective theory breaks down and it is no longer correct to identify the mesons as the physical degrees of freedom. Chiral symmetry is restored and you must work with a theory of quarks. Fortunately, QCD becomes (nearly) perturbative at that scale. I say nearly: charm quark physics ($m_c \sim 1$ GeV), and to a lesser extent bottom quark physics ($m_b \sim 5$ GeV), is quite difficult since you are technically outside the legitimate range of perturbative QCD *and* the chiral Lagrangian! There you need new effective field theories to describe the physics. This is the origin of such EFTs as Heavy Quark Effective Theory; Soft Collinear Effective Theory; and Nonrelativistic QCD. All of these theories have their own subtleties, but they all still obey the basic principles we worked out here. I will not mention them again, but encourage you to look at them on your own. A great starting point is [10].

Aside: One can repeat the same analysis for the standard model of particle physics without a Higgs boson, which is the same kind of model that we have discussed here. In that case $f = v \sim 200$ GeV, and NDA tells us $\Lambda_{\text{EWSB}} \sim 2$ TeV. This is called the electroweak scale. This tells us that if there is no Higgs boson, then there *must* be new physics at or below this scale. This is why the LHC is guaranteed to be a worthwhile experiment.

2.4 Power Counting

Now I would like to leave the chiral Lagrangian behind, and wrap up this lecture with a technique that is generic to any effective field theory calculation. We have seen that the power of EFT is that it keeps only the operators one needs to compute a physical process to a given order of accuracy. The method used to keep track of the "relevance" of operators is called **power counting**. We will have more to say about this later in the course, but let us see how one goes about doing it in the simplest cases.

[6] You can incorporate the ρ meson into the chiral Lagrangian if you wish – it is a vector meson, so you imagine it has a gauge symmetry associated with it, *etc.* Doing this improves the story somewhat, but it is of limited use in practice.

The idea is to assign a field a given **scaling**

$$\phi(x) \to \lambda^d \phi(x\lambda^{d_x}) \ , \tag{2.37}$$

where d, d_x are chosen wisely. Typically, for example, you would like to choose d so the the free action is invariant under the rescaling. d is called the **scaling dimension** of the field.

Notice that since the action is an integral over D dimensional spacetime, it implies that the free *Lagrangian* should scale like

$$\mathcal{L} \to \lambda^{-Dd_x} \mathcal{L} \ , \tag{2.38}$$

assuming all the x^μ scale the same way; that is, the theory is Lorentz covariant[7].

Consider a scalar field. It is a straightforward exercise to show, using $\partial_\mu \to \lambda^{-d_x}\partial_\mu$ that if we want (2.37)–(2.38) to hold, we need

$$d = \frac{2-D}{2} d_x \ . \tag{2.39}$$

In a Lorentz covariant theory, it is standard to chose $d_x = -1$, so the scaling dimension corresponds to the mass dimension and ordinary dimensional analysis; but this choice is certainly not unique, nor is it always the best choice to use. Deciding what to chose for d_x, and deciding whether one should treat all the x^μ the same way, is a nontrivial decision about the nature of the system you are trying to describe with your EFT.

Once you have determined the scaling dimension of every field, you can proceed to determine the scaling dimensions of every local operator[8], which can always be written down as a product of N fields:

$$\mathcal{O} = \prod_{n=1}^{N} \phi_n \ , \qquad d_{\mathcal{O}} = \sum_{n=1}^{N} d_n \ , \tag{2.40}$$

where ϕ_n need not be scalar fields in general, of course. Since these operators can appear in the action, they also come with coefficients that enforce the scaling:

$$S = \int d^D x \ c_i \mathcal{O}_i \Rightarrow \boxed{d_{c_i} = -(Dd_x + d_{\mathcal{O}_i})} \tag{2.41}$$

This (with $d_x = -1$) leads to an amazing theorem:

[7]This certainly does not need to be the case. For example in any nonrelativistic EFT, or in SCET where there is a preferred direction.

[8]Nonlocal operators are trickier, and we will not consider them at all in this course.

Theorem 9 *If $d_{\mathcal{O}_i} > 0$, which is always the case for local operators, then* $\boxed{d_{c_i} \leq D}$.

- Operators with $d_{\mathcal{O}_i} < D$ ($d_{c_i} > 0$) are called **relevant**.
- Operators with $d_{\mathcal{O}_i} = D$ ($d_{c_i} = 0$) are called **marginal**.
- Operators with $d_{\mathcal{O}_i} > D$ ($d_{c_i} < 0$) are called **irrelevant**.

We can now state the rule for constructing an EFT:

> You must keep **all** relevant and marginal operators in your theory, plus any number of irrelevant operators up to a fixed scaling dimension, depending on the level of accuracy you are aiming for.

This brings up an important and perhaps surprising point: **relevant operators are BAD!** They are big trouble, because we cannot forsake them! But irrelevant operators are great, because we have much more control over how important they are. Fortunately for us, Theorem 9 tells us that unless there are fields with $d = 0$, there are only a finite number of relevant operators. The famous counterexample to this is scalar field theory in $D = 2$. However, strange things happen there. These theories are conformal, and EFT fails us! We will not speak of this again, but it is important to remember that there are limits to this formalism.

An amusing point: this is probably *exactly* the opposite of what you are used to thinking – after all, irrelevant operators are "nonrenormalizable." But in fact, these are the *good* operators! We will talk much more about these points when we discuss the renormalization group toward the end of the course.

> **Aside:** In the standard model of particle physics, it is also amusing to notice that there is only one relevant operator that you can write down: the Higgs mass. All other operators are marginal or, if you allow yourself to give up on renormalizability, irrelevant. This is again strongly related to the naturalness problem of the Higgs sector.

2.5 Appendix: Some notes on spinors

It has come to my attention that several students are not comfortable with the conventions of two-component spinors. This is *very* important and useful, so let me briefly discuss the conventions here. For a much better and in depth discussion, take a look at [9].

Recall that in $D = 4$, there are two types of Weyl spinors:

- $(\frac{1}{2}, 0)$ are named "Left-handed spinors" and describe *left handed paritlces* and *right handed antiparticles.*

- $(0, \frac{1}{2})$ are named "Right-handed spinors" and describe *right handed paritlces* and *left handed antiparticles.*

That one spinor describes *both* modes is required by the CPT theorem.

It has become convention to **ALWAYS** use left handed spinors – that is, fields that transform in the $(\frac{1}{2}, 0)$ representation of the Lorentz group. Notice that if ψ is a left handed spinor, then $\overline{\psi}$ is a right handed spinor, where in two component notation, you can think of an overline as a dagger (there is no γ^0 like in four component notation).

With this convention in mind, we have the following rule:

- $\psi_L \equiv$ LH spinor describing a LH particle + CPT conjugate.
- $\psi_R \equiv$ **LH** spinor describing a LH **anti**particle + CPT conjugate.

So in two component notation, you should think of L, R as describing "particle" or "antiparticle," rather than chirality.

To translate this into four component notation, ψ_L and ψ_R pair up to form a Dirac spinor:

$$\Psi_D \equiv \begin{pmatrix} \psi_L \\ \overline{\psi}_R \end{pmatrix} \qquad \overline{\Psi}_D = \begin{pmatrix} \psi_R & \overline{\psi}_L \end{pmatrix} . \tag{2.42}$$

Notice how the lower component is $\overline{\psi}_R$ – that is *very* important! Then we can work out some spinor products:

$$\overline{\Psi}_D\Psi_D = \psi_L\psi_R + \overline{\psi}_L\overline{\psi}_R \ , \tag{2.43}$$

$$\overline{\Psi}_D\gamma^\mu\Psi_D = \psi_R\sigma^\mu\overline{\psi}_R + \overline{\psi}_L\overline{\sigma}^\mu\psi_L \ , \tag{2.44}$$

$$\vdots \tag{2.45}$$

This convention is used everywhere, especially among the SUSY community. It is very powerful, and very useful, since it eliminates the need to keep track of things like γ^5 and $\mathcal{C}$, the "charge conjugation matrix."

Finally, in Dirac notation, we have

$$\Psi_L = \frac{1}{2}(1-\gamma^5)\Psi_D = \psi_L \ , \tag{2.46}$$

$$\Psi_R = \frac{1}{2}(1+\gamma^5)\Psi_D = \overline{\psi}_R \ \ !! \tag{2.47}$$

Again, notice the conjugation on the right handed field! Therefore, in order to translate spinors from two to four component notation we need the rule:

$$\boxed{\begin{array}{ccc} \Psi_L & \leftrightarrow & \psi_L \\ \Psi_R & \leftrightarrow & \overline{\psi}_R \end{array}}$$

Finally, let me say something about index notation of two-spinors. There is a useful mnemonic where all $(\frac{1}{2},0)$ spinors come with a plain index ψ^α while all $(0,\frac{1}{2})$ spinors come wtih a *dotted* index $\overline{\psi}_{\dot\alpha}$. Indices are always contracted with the two component Levi-Civita symbol[9] $\varepsilon_{\alpha\beta}$ or $\varepsilon^{\dot\alpha\dot\beta}$. Notice that these symbols always come with the same type of index. When manipulating these tensors, the following identities may be useful:

$$\varepsilon_{ij}\varepsilon^{jk} = -\delta_i^k \ , \tag{2.48}$$

$$\varepsilon_{ij}\varepsilon^{kl} = \delta_i^k\delta_j^l - \delta_i^l\delta_j^k \ . \tag{2.49}$$

You can take a product of a left and right spinor, and this will give you a new object that has a dotted and plain index, in the $(\frac{1}{2},\frac{1}{2})$ representation. This is a vector, and can

[9] To see this, think about how you construct a singlet state out of two spin $\frac{1}{2}$ degrees of freedom in quantum mechanics. Also, think about the Young tableau.

be given an explicit vector index using the Pauli matrices $\sigma^{\mu}_{\alpha\dot{\beta}}$ and $\overline{\sigma}^{\mu}_{\dot{\alpha}\beta}$ (notice the order of spinor indices). These obey the useful identities

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\overline{\sigma}^{\dot{\beta}\beta}_{\mu} = 2\delta^{\alpha}_{\beta}\delta^{\dot{\alpha}}_{\dot{\beta}} , \tag{2.50}$$

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\sigma_{\mu\beta\dot{\beta}} = 2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} . \tag{2.51}$$

Combined with the fact that the Weyl fields obey a Grassman (anticommuting) algebra, we have some useful identities

$$\xi\chi = \chi\xi , \qquad \overline{\xi}\chi = \overline{\chi}\overline{\xi} ,$$

$$\overline{\xi}\overline{\sigma}^{\mu}\chi = -\chi\sigma^{\mu}\overline{\xi} , \qquad \xi\sigma^{\mu}\overline{\sigma}^{\nu}\chi = \chi\sigma^{\nu}\overline{\sigma}^{\mu}\xi .$$

I leave the proofs of these to you. From these, we can also prove the two-component Fierz Identities [9]

$$(\chi\xi)(\theta\phi) = -(\chi\theta)(\phi\xi) - (\chi\phi)(\xi\theta) , \tag{2.52}$$

$$(\theta\sigma^{\mu}\overline{\chi})(\overline{\xi}\overline{\sigma}_{\mu}\phi) = -2(\theta\phi)(\overline{\chi}\overline{\xi}) , \tag{2.53}$$

$$(\theta\sigma^{\mu}\overline{\chi})(\xi\sigma_{\mu}\overline{\phi}) = +2(\theta\xi)(\overline{\chi}\overline{\phi}) . \tag{2.54}$$

Chapter 3

What Quantum Does to Symmetries

Everything that we have been doing so far has assumed that if we have a symmetry at the classical level, then it is there at the quantum level as well. If it is broken spontaneously we know what to do, and if it is broken explicitly we can still get far using the technique of spurion analysis. But what if the symmetry is not broken by the vacuum or an explicit operator, but by the quantum effects themselves? When this happens, we say that the symmetry is anomalous, or that the symmetry is "dynamically broken." In this lecture, we will consider how this can happen, and what it does to our effective theory.

3.1 Anomalies in the Path Integral

At the quantum level, a typical (perturbative) QFT is defined by an action (dynamics), a vacuum, *and* a regulator. This last item is vital – QFT perturbation theory *requires* a regulator to extract finite resuts, except for very special circumstances. Furthermore, one regulator does not give you the same answer as another regulator. While it is true that *physical* results must of course by regulator independent, the perturbative QFT itself is not. With this in mind, we can make a definition:

Definition 16 *A symmetry is* **anomalous** *if it is realized in the action either linearly or nonlinearly, but is violated by the regulator – for all regulators.*

The last point about the symmetry breaking happening for every regulator is very important, as can be shown by a famous counterexample. Consider QED with momentum cutoff. It is well known that the momentum cutoff breaks the gauge symmetry, since it

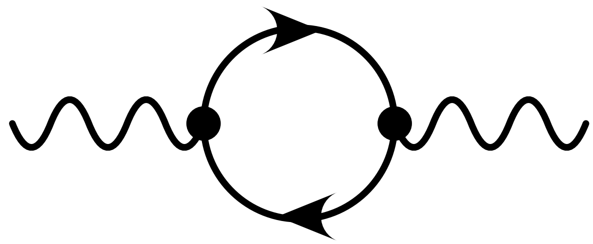


Figure 3.1: Photon self energy diagram in QED.

removes some of the momentum modes from the theory so gauge transformations cannot be completely cancelled. Sure enough, at one loop you have a quadratically divergent photon mass from Figure 3.1

$$\Pi^{\mu\nu} \sim g^{\mu\nu}\Lambda^2 + \cdots \tag{3.1}$$

However, if instead of using momentum cutoff, you used Pauli-Villars or dimensional regularization, which *does* maintain the gauge symmetry, this no longer happens:

$$\Pi^{\mu\nu} = (g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi(q^2) \ , \tag{3.2}$$

where $\Pi(q^2)$ can only be logarithmically divergent at worst. So even though there is a particular regulator that breaks the gauge symmetry, this is not anomalous.

There are many ways to derive anomalies, but I have always thought that the path integral is the cleanest. Amusingly, it was thought for a long time that the existence of anomalies was the nail in the coffin for path integrals, since no one knew how to derive them. It was not until the late 1970s that Fujikawa [11] figured out the trick, and therefore path integral anomalies are sometimes called "Fujikawa anomalies."

Recall that if you have a set of fields ϕ^a, not necessarily scalars, and an action on these fields $S[\phi]$, the path integral is given by

$$Z[J] \equiv \int \mathcal{D}\phi \ e^{iS[\phi(x)]+i\int d^dx J(x)\phi(x)} \ , \tag{3.3}$$

and in principle, all the physics, both classical and quantum, is contained in Z, analogous to the partition function in statistical mechanics.

Now recall Hamilton's principle from classical field theory:

Under variations of ϕ : $S[\phi_0 + \delta\phi] = S[\phi_0]$, where ϕ_0 is the solution to the classical field equations.

To see how this is generalized to QFT, consider what happens to the path integral under such a variation:

$$Z \to \int \mathcal{D}\phi \; e^{iS[\phi+\delta\phi]} = \int \mathcal{D}\phi \; e^{iS[\phi(x)]} \left[1 + \left(i\frac{\delta S}{\delta\phi} \right) \delta\phi + \cdots \right] , \tag{3.4}$$

where we have used that $\mathcal{D}(\phi + \delta\phi) = \mathcal{D}\phi$ since it is just a shift of coordinates. We will return to this point later.

So we have:

$$\begin{aligned} Z' &= Z + \int \mathcal{D}\phi \; e^{iS[\phi(x)]} \left(i\frac{\delta S}{\delta\phi} \right) \delta\phi \\ &= Z \left[1 + \left\langle \left(i\frac{\delta S}{\delta\phi} \right) \delta\phi \right\rangle \right] . \end{aligned} \tag{3.5}$$

Now since the path integral is independent of ϕ – it is only a dummy variable, after all – $Z = Z'$ for an arbitrary choice of $\delta\phi$, and so

$$\left\langle \left(i\frac{\delta S}{\delta\phi} \right) \right\rangle = 0 . \tag{3.6}$$

This is just the correspondence principle at work! QFT people call this (and its generalization with more operator insertions) the *Schwinger-Dyson Equations*. See [7] for a more in depth discussion of these equations.

Using the same techniques, we can derive the QFT version of Noether's theorem:

Theorem 10 *Assume that there is a classical symmetry of the action:*

$$S[\phi + \delta\phi] = S[\phi] ,$$

where unlike before, this is true for all ϕ, not just a solution of the field equations, and $\delta\phi$ is the symmetry transformation, which we parametrize as $\delta\phi \equiv \epsilon\Delta\phi$. Then

$$\boxed{\delta S[\phi] = \int d^d x \; \epsilon \partial_\mu j^\mu} , \tag{3.7}$$

where j^μ is the Noether current.

Now let us see what happens in the path integral:

$$Z \to \int \mathcal{D}\phi \; e^{iS[\phi] + \int d^d x \; \epsilon(\partial_\mu j^\mu)} \tag{3.8}$$

Since this is supposed to be a symmetry, Z cannot depend on ϵ, so

$$\frac{1}{Z}\frac{\delta Z}{\delta\epsilon} = \langle\partial_\mu j^\mu\rangle = 0 \ . \tag{3.9}$$

So once again, we recover Noether's theorem through the correspondence principle.

However, there is a big flaw in this argument: I have been very cavalier about the behavior of the measure during all of these changes of coordinates. Indeed, the measure generally transforms as

$$\mathcal{D}\phi' = \mathcal{D}\phi \ \mathcal{J}[\phi] \ , \tag{3.10}$$

where

$$\mathcal{J}[\phi] \equiv \det\left(\frac{\delta\phi'}{\delta\phi}\right) = e^{\mathrm{Tr}\log\left(\frac{\delta\phi'}{\delta\phi}\right)} = e^{\mathrm{Tr}\log(1+i\epsilon C)} = e^{i\mathrm{Tr}(\epsilon C)} \ . \tag{3.11}$$

and $\phi' = (1+i\epsilon C)\phi$ is an infinitesimal linear transformation. So in fact, we really have

$$Z' = \int \mathcal{D}\phi \ e^{iS[\phi]+\int d^dx \ \epsilon(\partial_\mu j^\mu + \mathrm{tr}(C))} \tag{3.12}$$

and Noether's theorem becomes

$$\frac{1}{Z}\frac{\delta Z}{\delta\epsilon} = \langle\partial_\mu j^\mu + \mathrm{tr}(C)\rangle = 0 \ . \tag{3.13}$$

If tr $C = 0$ then nothing has changed. But if tr $C \neq 0$ then *this* is the anomaly equation. Classically, tr C will vanish, but once you go to the quantum theory, it might not. In particular – the regulator might cause it to develop a finite piece [11].

3.2 The Chiral Anomaly and It's Consequences

The classic example of an anomaly, and the one I will be discussing in this lecture, is the **chiral anomaly**. Recall that massless fermions have two (classically) conserved currents:

$$\begin{aligned} j^\mu &= \overline{\Psi}\gamma^\mu\Psi = \psi_R\sigma^\mu\overline{\psi}_R + \overline{\psi}_L\overline{\sigma}^\mu\psi_L \ , & (3.14) \\ j_5^\mu &= \overline{\Psi}\gamma^\mu\gamma^5\Psi = \psi_R\sigma^\mu\overline{\psi}_R - \overline{\psi}_L\overline{\sigma}^\mu\psi_L \ . & (3.15) \end{aligned}$$

j^μ is the fermion number current that couples to gauge bosons. If it is anomalous, then the gauge symmetry is dynamically broken. This is a disaster – it would signal a breakdown of unitarity, and is definitely something we do not want. However, j_5^μ is the "axial number" current and it can do whatever it wants! It is well known that it is anomalous, and the anomaly equation is

$$\partial_\mu j_5^\mu = -\frac{g^2 T(R)}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \ , \tag{3.16}$$

for an $SU(N)$ theory coupled to (massless) fermions in the representation R.

As always, it is convenient to work in two component notation and define

$$j_L^{\mu a} = \overline{\psi}_L\overline{\sigma}^\mu T^a \psi_L \ , \tag{3.17}$$

$$j_R^{\mu a} = \psi_R \sigma^\mu T^a \overline{\psi}_R \ , \tag{3.18}$$

so that $j^\mu = j_R^\mu + j_L^\mu$ and $j_5^\mu = j_R^\mu - j_L^\mu$. Then in general, any *chiral theory*[1] will have an anomaly in both j_L^μ and j_R^μ such that the anomaly cancels in the sum and adds in the difference:

$$\partial_\mu j_L^{\mu a} = +\frac{g^2\mathcal{A}^{abc}}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F^b_{\mu\nu}F^c_{\rho\sigma} \ , \tag{3.19}$$

$$\partial_\mu j_R^{\mu a} = -\frac{g^2\mathcal{A}^{abc}}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F^b_{\mu\nu}F^c_{\rho\sigma} \ , \tag{3.20}$$

where $\mathcal{A}$ is a new group theory index object that we must try to work out. To compute it, consider the correlation function of three L currents $C(x,y,z) = \langle j_L^{\mu a}(x) j_L^{\nu b}(y) j_L^{\rho c}(z)\rangle$. To leading order, this correlator is given by the diagrams in Figure 3.2.

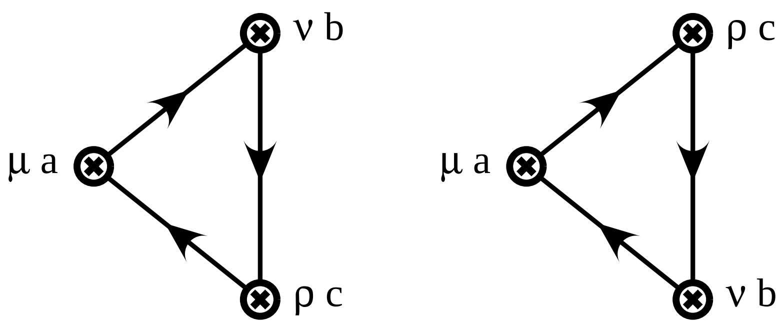


Figure 3.2: Triangle diagrams that are generated by the three point correlation function of chiral currents.

Let us compute the group theory structure. Since they involve closed loops, there is a trace over generators. We have for fermions in representation R

$$C \sim \mathrm{Tr}[T_R^a T_R^b T_R^c] + \mathrm{Tr}[T_R^a T_R^c T_R^b] = \mathrm{Tr}\{T_R^a [T_R^b, T_R^c]_+\} \tag{3.21}$$

[1]A **chiral theory** is any theory of fermions that treat left and right fermions differently. This is the opposite of a vectorlike theory, which treats left and right fermion fields the same way. Agan, remember that "left" and "right" really mean "particle" and "antiparticle" in this language.

But this is nothing more than the expression for the anomaly coefficient (1.14). This is where it gets its name. In addition, remember that if $\{T_1^a\}$ generate the group G_1 and $\{T_2^a\}$ generate the group G_2, then the traces factorize $\mathrm{Tr}[T_1 \otimes T_2] = \mathrm{Tr}[T_1] \cdot \mathrm{Tr}[T_2]$.

Thus to compute anomalies, one must first compute the anomaly coefficients of the theory, for which there are tables and rules discussed at the end of the first lecture. In particular, there is the very important rule that $A(R) = -A(\overline{R})$. This means that a theory of *Dirac* fermions is always anomaly free! Also, if you have fermions that transform in a real representation, either vectorlike or something like the Adjoint representation, then there is no anomaly. Finally, only $SU(N)$ groups with $N \geq 3$ or $U(1)$ groups[2] can have anomalies, since they're the only groups with complex representations.

An amusing sidenote: in Figure 3.2 we can attach gravitons to two of the vertices (replacing the chiral currents with energy-momentum tensors), and then the anomaly would be proportional to $\mathrm{Tr}(T^a)$. This can be nonvanishing only for $U(1)$, and it leads to the important result that unless all the charges of the fields sum to zero, the theory has a "gravitational anomaly." This is why people will say, "gravity breaks global symmetries."

If the symmetries are global, anomalies can lead to interesting physics. For example, in the chiral Lagrangian, it is the dominant contribution to the decay of the π^0 to two photons. However, if we gauge the chiral symmetry, it is *very* important that there are no anomalies, otherwise there would be the usual story of unitarity breakdown. This is a very important constraint on model building. The standard model, for example, is anomaly free because of its funny "flavor structure" – see [7].

3.3 Anomalies and Instantons

Unfortunately, time ran out before I had a chance to give this lecture, but I'll include my notes here for possible future expansions.

[2]The generator for a $U(1)$ is just the charge matrix, so for anomalies to vanish, the charges, or products of charges, must sum to zero.

Chapter 4

Quantum Bible Stoires: "Death and Rebirth of Scale Invariance"

As we have seen earlier in Lecture 2, many actions have a natural "scale invariance," where you can define a "scaling dimension," which is naively just set from dimensional analysis[1], and keep track of the importance of various terms in the effective action by computing these dimensions. Even if scale invariance is not an exact symmetry, you can use a spurion analysis to keep track of symmetry breaking terms as well, such as mass terms. This leads to some very powerful theorems, analogous to the Ward Identities of gauge theories, that tell us a lot about the nature of correlation functions coming out of such a theory. And we will find that these theorems are *wrong*!

These lectures come almost directly out of Sidney Coleman's lecture given in Chapter 3 of [12].

4.1 "Low Energy" Theorems

To begin, let me remind you how the Ward Identity goes:

Theorem 11 *If ϕ are a set of fields that transform as $\delta\phi$ under a symmetry with a current*

[1] These numbers from dimensional analysis are also sometimes called the **engineering dimension** of the field or operator.

j^μ *where* $\partial_\mu j^\mu = \Delta$, *we have*

$$\begin{aligned}\frac{\partial}{\partial y^\mu}\langle j^\mu(y)\phi(x_1)\cdots\phi(x_n)\rangle &= \langle\Delta(y)\phi(x_1)\cdots\phi(x_n)\rangle \\ &\quad -i\delta^{(4)}(y-x_1)\langle\delta\phi(x_1)\cdots\phi(x_n)\rangle \\ &\qquad\vdots \\ &\quad -i\delta^{(4)}(y-x_n)\langle\phi(x_1)\cdots\delta\phi(x_n)\rangle\ . \end{aligned} \tag{4.1}$$

The **Low Energy Theorems** come from doing $\int d^4y$ of both sides of this identity:

$$\boxed{\int d^4y\langle\Delta(y)\phi(x_1)\cdots\phi(x_n)\rangle = i\langle\delta\phi(x_1)\cdots\phi(x_n)\rangle + \cdots +\ i\langle\phi(x_1)\cdots\delta\phi(x_n)\rangle} \tag{4.2}$$

This result is the basis for many other results, most famously the "partially conserved axial current (PCAC)" hypothesis of nuclear physics, where the current in question is the axial vector current discussed in the last lecture and Δ is the pion mass operator; then this expression leads to the celebrated "Goldberger-Trieman" relation. Notice that like the Optical Theorem, it is one of those exact, nonperturbative statements, and therefore must be used carefully in combination with perturbative statements.

4.2 Scaling Symmetry and Its Consequences

We would like to apply (4.2) to the scaling symmetry[2]

$$\delta\phi = (d + x^\mu\partial_\mu)\phi(x)\ , \tag{4.3}$$

where d is the **scaling dimension** of the field. For now, we will identify it with the engineering dimension mentioned above, but we will see that this is not right in general.

For an action to be scale invariant, at least classically, all operators must have scale dimension equal to the spacetime dimension D, so

$$\mathcal{L} \to \mathcal{L} + (D + x^\mu\partial_\mu)\mathcal{L}\ . \tag{4.4}$$

Then this will vanish in the action after integration by parts.

[2]Notice that $\delta\phi \equiv \phi'(x') - \phi(x)$.

Notice that we already see a problem, since you know that products of fields at the same spactime point are funny things in the quantum theory since these fields will not commute. But this aside, the action is still invariant, so long as $d(\mathcal{L}) = D$.

The corresponding scale current is given by

$$s^\mu \equiv x_\nu \Theta^{\mu\nu} \ , \tag{4.5}$$

where $\Theta^{\mu\nu}$ is the energy momentum tensor. That the current depends on Θ is not surprising, since scale symmetry is a spacetime symmetry. So for a scale invariant theory

$$\boxed{\partial_\mu s^\mu = \Theta^\mu_\mu = 0} \tag{4.6}$$

where we have used the conservation of momentum $\partial_\mu \Theta^{\mu\nu} = 0$.

> **Aside:** The energy momentum tensor in (4.5) is *not* the same as the symmetric energy-momentum tensor $T^{\mu\nu}$ that couples to classical gravity. This is a subtle but vitally important point! Recall that energy-momentum tensors can only be defined up to four divergences
>
> $$T_1^{\mu\nu} \cong T_2^{\mu\nu} + \partial_\alpha \Lambda^{\alpha\mu\nu} \ ,$$
>
> where $\Lambda^{\alpha\mu\nu} = -\Lambda^{\mu\alpha\nu}$ and $\Lambda^{\alpha\mu\nu} = -\Lambda^{\nu\mu\alpha}$. $\Theta^{\mu\nu}$ is a very particular choice of tensor where this freedom is removed.

Now let us consider the low energy theorem for scale invariance. For definiteness and simplicity, I will consider a (real) scalar field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m_0^2 \phi^2 - \frac{1}{4!}\lambda_0 \phi^4 \ , \tag{4.7}$$

where I included naughts on the couplings to remind you that at this stage I am working with the *un*renormalized action. Define $\Gamma^{(n)}(p_1, \ldots, p_n)$ to be the 1PI *renormalized* Green's functions with n external lines, and $\Gamma^{(n)}_\Delta(k; p_1, \ldots, p_n)$ as the 1PI renormalized Green's functions with n external lines *and* one insertion of $\Delta \equiv \partial_\mu s^\mu = m^2 \phi^2$, which in the case of a real scalar field and the scale current (notice that this is the *renormalized* mass). Then the low energy theorem (4.2) can be written as

$$\left[\sum_{i=1}^{n-1} p_i \cdot \frac{\partial}{\partial p_i} + nd - 4\right] \Gamma^{(n)}(p_1, \ldots, p_n) = -i\Gamma^{(n)}_\Delta(0; p_1, \ldots, p_n) \ . \tag{4.8}$$

The first two terms are the transformation law for a product of n fields in momentum space – the sum is only over $n-1$ of the momenta because of an overall momentum-conserving delta function; the 4 comes from the scaling of this residual delta function. Notice that the right hand side is evaluated at $k = 0$, hence the name "low energy theorem." The proof of this expression is a direct application of Fourier transforms to (4.2) and the definition of the 1PI coefficients from the effective action

$$\Gamma[\phi] = \int d^D x \sum_n \frac{1}{n!} \Gamma^{(n)} \phi^n \ .$$

It is useful to define a new variable

$$s = \sum_{i=1}^{n} p_i^2 \ ; \tag{4.9}$$

all kinematics can then be described in terms of the variable s and the dimensionless four-vector products $p_i \cdot p_j / s$. Then ordinary, boring dimensional analysis tells us

$$\Gamma^{(n)} = s^{(4-n)/2} F_n \left(\frac{s}{m^2}, \lambda, \frac{p_i \cdot p_j}{s} \right) \ , \tag{4.10}$$

where again, λ, m^2 are the renormalized coupling and mass, and F_n is a dimensionless function. Now using the chain rule and rewriting terms, we can write

$$\boxed{\left[m \frac{\partial}{\partial m} + n(1-d) \right] \Gamma^{(n)}(p_1, \ldots, p_n) = -i \Gamma_\Delta^{(n)}(0; p_1, \ldots, p_n)} \tag{4.11}$$

This is the version of the low energy scale theorem that we will be studying.

Now I would like to introduce a very powerful theorem from QFT called the **Weinberg Power Counting Theorem**[3]

Theorem 12 *In the "deep Euclidean regime" (DER) where* $s \to \infty$, $p_i \cdot p_j / s =$*constant, we have*

$$\Gamma^{(n)} \longrightarrow s^{(4-n)/2} \ f \left(\log \left(\frac{s}{m^2} \right) \right) \ , \tag{4.12}$$

$$\Gamma_\Delta^{(n)} \longrightarrow s^{(2-n)/2} \ \tilde{f} \left(\log \left(\frac{s}{m^2} \right) \right) \ . \tag{4.13}$$

The $f, \tilde{f}$ *are polynomials in the log, and their coefficients depend on the coupling constant* λ *and the (constant) kinematic variables* $p_i \cdot p_j / s$.

[3] Actually, this is only a small corollary to the full power counting theorem, but it's enough for our purposes.

You can think of the DER as the limit that the energy goes to infinity with the scattering angles held fixed. Γ_Δ has a lower power of s since the mass insertion replaces a p^2 somewhere. This means that in the DER, $\Gamma \gg \Gamma_\Delta$. Combining this with our low energy theorem (4.8) and using $d = 1$ gives

$$m\frac{\partial}{\partial m}\Gamma^{(n)} = 0 \ , \tag{4.14}$$

and so $\Gamma^{(n)}$ has no $\log(s/m^2)$ to *all orders in perturbation theory!*

This is amazing! This is incredible! And this is wrong!! Anyone who has done a one loop calculation in ϕ^4 theory knows that the $2 \to 2$ scattering amplitude has logarithms in it. So we conclude that scale invariance must be an *anomalous* symmetry!

> **Aside:** It was not necessary to go to the DER – just easier. You might worry that doing this is rather silly, and that it is not surprising things break down in a region of phase space that is not even physical. The counterargument to this goes as follows: low energy theorems are valid for any value of coupling constant and momenta, including the DER. So to show that it doesn't work *anywhere*, it is enough to show that it doesn't work *somewhere*. Coleman gets into this more in [12], and I encourage you to go through his argument carefully.

4.3 Fixing the Scaling Anomaly I: Anomalous Dimensions

Recall the chiral anomaly from the previous lecture

$$\partial_\mu j_5^\mu = -\frac{g^2 T(R)}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \equiv \partial_\mu K^\mu \ . \tag{4.15}$$

It is well known that the right hand side of this anomaly equation is a total divergence. This means that adding this (naively renormalizable) operator to the action will nave no effect on perturbative calculations. However, it also means that while j_5^μ is not conserved, the quantity $j_5^\mu - K^\mu$ *is* conserved, although it generally is not gauge invariant! Thus the axial anomaly is resolved as long as we redefine what "axial charge" is. This leads us to the famous Atiyah-Singer index theorem.

So you might think that perhaps the same thing will happen with the scale anomaly. Let us redefine the scaling dimension (d_s) of a field (or operator) as

$$d_s = d_e + \gamma \tag{4.16}$$

where d_e is the engineering dimension that comes from dimensional analysis, and γ is a new number called the **anomalous dimension** that represents the correction to scaling that comes from the anomaly.

> **Aside:** γ plays the same role as the critical exponent η in condensed matter. From ordinary dimensional analysis,
>
> $$\langle\phi(x)\phi(y)\rangle = \frac{c}{r^{2d_e}} .$$
>
> If there are no dimensionful couplings, you might think that there is nothing more to be said. But if there is a distance scale r_0, then $c = c_0(r/r_0)^{-2\gamma}$ and these contributions do not follow from dimensional analysis. This is the origin of the anomalous dimension.
> In fact, even if there is no such scale at the classical level, there is always the cutoff at the loop level (in condensed matter, this is the lattice spacing). So there can be additional powers of r that appear in the correlation functions beyond the ordinary dimensional analysis story. This is why it is indeed an anomaly!

The anomalous dimension may be small (perturbative) or large – at this point it doesn't matter since the low energy theorem is a nonperturbative statement. If this was all there was to it, then like the chiral anomaly, we would be in great shape!

Unfortunately, this still does not work. To see this, let us reconsider our example of massive ϕ^4 theory. Using (4.8) with $d \neq 1$ we still find

$$m\frac{\partial}{\partial m}\left(\frac{\Gamma^{(4)}}{[\Gamma^{(2)}]^2}\right) = 0 . \tag{4.17}$$

Thus, while $\Gamma^{(2)}$, $\Gamma^{(4)}$ can have logarithms, the above ratio does not. Is this true? Once again, let us go to the DER. A straightforward calculation (done in [7]) gives

$$\Gamma^{(2)} \sim s + \mathcal{O}(\lambda^2) , \tag{4.18}$$

$$\Gamma^{(4)} \sim \lambda + a\lambda^2 \log(s/m^2) + b\lambda^2 . \tag{4.19}$$

Again, a and b can depend on $p_i \cdot p_j/s$ but are constants in the DER. Taking the ratio above gives

$$\frac{\Gamma^{(4)}}{[\Gamma^{(2)}]^2} \sim \frac{\lambda}{s^2} + \frac{a\lambda^2}{s^2}\log(s/m^2) + b\frac{\lambda^2}{s^2} + \mathcal{O}(\lambda^2) . \tag{4.20}$$

Once again, we find a logarithm[4] that the low energy theorem tells us should not be there.

[4] At first glance, you might think that this term is not there, since as $s \to \infty$ these terms vanish. However, it is a straightforward exercise to show that if you carefully keep higher order terms in s^{-1} throughout the calculation, you cannot cancel this logarithm and so this is a real violation of (4.17).

So unlike the chiral anomaly, the scale anomaly does not go away so easily.

4.4 Fixing the Scaling Anomaly II: Running Couplings

Let us go back and try to figure out what went wrong. The best way to proceed is to go back to the *un*renormalized theory. Recall

$$\phi_0 = Z_\phi^{1/2}\phi \quad \Rightarrow \quad \Gamma^{(n)} = Z_\phi^{n/2}\Gamma_0^{(n)} \ . \tag{4.21}$$

Also recall that $\Gamma_\Delta^{(n)} = ZZ_\phi^{n/2}\Gamma_{0\Delta}^{(n)}$ for some Z such that $\Gamma_\Delta^{(n)}$ is finite. Putting this together with the low energy theorem we have

$$m_0\frac{\partial}{\partial m_0}\Gamma_0^{(n)}(p_i) = i\Gamma_{0\Delta}^{(n)}(0;p_i) \ . \tag{4.22}$$

Multiplying both sides by $ZZ_\phi^{n/2}$ and inserting $1 = (Z_\phi/Z_\phi)^{n/2}$ inside the derivative, we find

$$i\Gamma_{0\Delta}^{(n)}(0;p_i) = Zm_0\frac{\partial}{\partial m_0}\Gamma^{(n)}(p_i) - \frac{n}{2}Zm_0\frac{\partial \log Z_\phi}{\partial m_0}\Gamma^{(n)}(p_i) \ . \tag{4.23}$$

This is not quite what we want, since $\Gamma^{(n)}$ is only finite when expressed in terms of m and not m_0, so using the chain rule, we get

$$i\Gamma_{0\Delta}^{(n)}(0;p_i) = \left[\left(Zm_0\frac{\partial m}{\partial m_0}\right)\frac{\partial}{\partial m} + \left(Zm_0\frac{\partial \lambda}{\partial m_0}\right)\frac{\partial}{\partial \lambda} - n\left(\frac{Zm_0}{2}\frac{\partial \log Z_\phi}{\partial m_0}\right)\right]\Gamma^{(n)}(p_i) \ . \tag{4.24}$$

Notice that the only possible cutoff dependence lives inside the three functions inside the parentheses on the right hand side of this equation. However, they are all constants, independent of momentum or n, so that by ordinary dimensional analysis, they can only depend on λ. Choose the finite part of Z so that the first term in parentheses is just m, the renomalized mass, and calling the second and third terms in parentheses β and γ, respectively, we can finally state the full low energy theorem for scaling symmetry:

$$\boxed{\left[m\frac{\partial}{\partial m} + \beta(\lambda)\frac{\partial}{\partial \lambda} - n\gamma(\lambda)\right]\Gamma^{(n)}(p_i) = i\Gamma_{0\Delta}^{(n)}(0;p_i)} \tag{4.25}$$

This equation is known as the **Callan-Symanzik Equation**, and it is *this* equation that correctly describes scaling.

There are some things to say about this equation:

- If $\beta \equiv 0$, then indeed, we are as we were with the chiral anomaly – scaling dimensions are shifted from the engineering dimension by a constant γ, but the low energy theorems from earlier still hold. However, the cases where $\beta \equiv 0$ are rare, and correspond to the **conformal field theories**.

- The scaling low energy theorem is quite useless in general! For the chiral case, we were able to use the low energy theorems to relate one $\Gamma^{(n)}$ to another. *If* $\beta \equiv 0$, then we could still do that, such as (4.17). These are the **conformal Ward identities** discussed in [13], for example. However, we have lost that power here. Only if you knew β exactly could you hope to do this. But if you know β (and γ) exactly, then you have solved the entire problem, and you are wasting your time with low energy theorems!

- In the DER, we can ignore $\Gamma^{(n)}_{\Delta}(0; p_i)$ and the Callan-Symanzik equation becomes the **renormalization group equation**. Its solution in terms of $t \equiv \frac{1}{2}\log(s/m^2)$ is well known:

$$\Gamma^{(n)} = s^{(4-n)/2}\ F^{(n)}(\overline{\lambda}(\lambda, t), p_i \cdot p_j/s)\ e^{-n\int_{t_0}^{t} dt' \gamma(\overline{\lambda}(\lambda,t))}\ , \tag{4.26}$$

where

$$\frac{d\overline{\lambda}}{dt} = \beta(\overline{\lambda})\ , \tag{4.27}$$

$$\overline{\lambda}(\lambda, t_0) = \lambda\ , \tag{4.28}$$

and $F^{(n)}$ is an unknown function of the remaining variables. So up to the "running coupling" effects, we have the same scaling prediction from the low energy theorems. This tells us that *the RG equations are only valid in the DER!* There are ways around this, however. See [12], chapter 3.4 for more details.

Chapter 5

The Coleman-Weinberg Effective Potential

Theories of scalar fields are very important to a theorist. From a model building point of view, they are the simplest theories you can write down, so they are always a good place to start. Also, if a theory is Lorentz invariant, the effective degrees of freedom can often be modeled by effective scalar fields. This is what happens, for example, in theories of the inflaton, the Higgs mechanism, *etc.* Also, scalar fields appear naturally in supersymmetric theories, or in theories of spontaneous symmetry breaking.

In this lecture we will discuss the behavior of theories of scalar fields. The primary reference that I'll be following very closely is the must read paper by Sidney Coleman and Erick Weinberg [14]. Most of the important material, and a lot of other useful bits of information, can also be found in Coleman's *Aspects of Symmetry*, Chapter 5 [12].

The million dollar question is: *Can we have a situation where the quantum effects push the couplings around in such a way that we generate a vacuum where the symmetry is spontaneously broken, even if there is no such vacuum at tree level?*

5.1 Behavior of the Effective Action

We talked about how if a scalar potential $V(\phi)$ is such that $\langle\phi\rangle \neq 0$ when ϕ is charged under a symmetry group G, then G is spontaneously broken. For example,

$$V_0(\phi) = m^2|\phi|^2 + \lambda|\phi|^4 \quad \Rightarrow \quad \langle\phi\rangle = 0,\ \pm\sqrt{\frac{-m^2}{2\lambda}}\ . \tag{5.1}$$

So if $m^2 < 0$ there is spontaneous symmetry breaking. But implicit in all of this logic was the statement that m^2, λ are the *renormalized* mass and coupling. If (5.1) is the tree level result, we know that at higher order we will have

$$V(\phi) = \overline{m}^2|\phi|^2 + \overline{\lambda}|\phi|^4 + \cdots \tag{5.2}$$

where $\overline{m}^2, \overline{\lambda}$ are radiatively corrected, and the ellipses refer to higher dimensional operators that are generated with finite coefficients, since V_0 is renormalizable.

This leads to a question: Can we have a situation where $V_0(\langle\phi\rangle)$ has a minimum at $\langle\phi\rangle = 0$, but $V(\langle\phi\rangle)$ has a minimum at $\langle\phi\rangle \neq 0$? And if so, what are the effective mass and couplings?

To understand the answers to these questions, it is best to work with the effective action

$$\Gamma[\phi_c] = W[J] - \int d^4x\ J(x)\phi_c(x) \tag{5.3}$$

This can be thought of as the "Gibbs Free Energy" of the field theory. Previously we used the expansion of Γ in terms of 1PI functions, but now we're going to do a different expansion – the **momentum (or derivative) expansion**:

$$\Gamma[\phi_c] = \int d^4x[-V_{\text{eff}}(\phi_c) + \frac{1}{2}Z(\phi_c)(\partial_\mu\phi_c)^2 + \cdots] \tag{5.4}$$

In particular, if we study $\phi_c(x) \equiv \phi_c$ (constant), then only the first term contributes to the effective action. This term without the volume integral, is called the **effective potential**.

Now in our case, we have the effective potential in (5.2), where

$$\overline{m}^2(\mu) \equiv \left.\frac{d^2V_{\text{eff}}}{d\phi_c^2}\right|_{\phi_c=\mu} \tag{5.5}$$

$$\overline{\lambda}(\mu) \equiv \left.\frac{d^4V_{\text{eff}}}{d\phi_c^4}\right|_{\phi_c=\mu} \tag{5.6}$$

These are our renormalization conditions. There are a few things to say at this stage:

- These renormalization conditions are a little funny: they are defined at values of ϕ_c, *not* at p^2 like usual, since now $p^2 = 0$. This distinction will be important in what follows.

- As long as there are no IR divergences, which now appear as $\phi_c \to 0$, we can safely set $\mu = 0$ in the renormalization conditions above. But in general, $\mu \neq 0$, and as we will be interested in theories with IR divergences, we will keep it explicit in what follows.

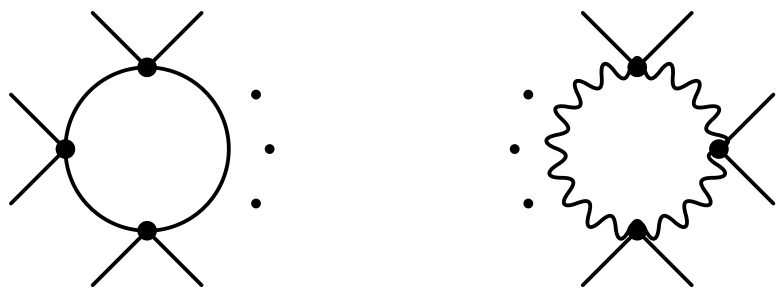


Figure 5.1: One loop diagrams. (a) Scalar loops. (b) Vector loops.

5.2 A Simple (perhaps *too* simple) Example

Consider a *massless* real scalar field theory:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4!}\phi^4 + \frac{1}{2}A(\partial_\mu\phi)^2 - \frac{1}{2}B\phi^2 - \frac{1}{4!}C\phi^4 \ , \tag{5.7}$$

where A, B, C are counterterms.

Now let us perform the **loop expansion** on this theory. The way this works is as follows: imagine that the action scales with a quantity a, so $S[\phi, a] = S[\phi]/a$. Then it is clear that every propagator in a diagram gives a factor of a and every vertex gives a factor of a^{-1}, so every diagram goes like a^{I-V}, where I is the total number of internal lines, and V is the total number of vertices. But from elementary graph theory, $L - I + V = 1$; therefore every graph scales like a^{L-1}, so a loop expansion is like an expansion in a.

The one loop diagrams from (5.7) are particularly straightforward, they are given generally by Figure 5.1a. They always take the form of a single loop with an arbitrary number of ϕ^2 insertions. Summing over all such insertions gives

$$\Delta V(\phi_c) = i\int \frac{d^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{\frac{1}{2}\lambda\phi_c^2}{k^2 + i\varepsilon}\right)^n \ . \tag{5.8}$$

Some words about the factors:

- The factor of i out in front is from the definition of the path integral.
- The $\frac{1}{2n}$ is a combinatoric factor – rotations ($\frac{1}{n}$) and reflections ($\frac{1}{2}$) of the loop do not change the diagram, so the $\frac{1}{n!}$ in Dyson's formula is not completely cancelled.

- The $\frac{1}{2}$ in the numerator is an extra bose factor, since exchanging two external lines at the same vertex does not give a new diagram, so the $\frac{1}{4!}$ in the vertex is not completely cancelled.

This sum is straightforward, and after Wick rotating we find up to an irrelevant constant

$$\Delta V = \frac{1}{2}\int \frac{d^4k_E}{(2\pi^4)} \log\left(k_E^2 + \frac{1}{2}\lambda\phi_c^2\right) \ . \tag{5.9}$$

Notice that this integral is IR divergent as $\phi_c \to 0$. This is *exactly* what we expected would happen. But as long as we keep away from the origin in field space, we are okay. Also notice that it is UV divergent, but this should not bother anyone. You can regulate any way you want. Let us use momentum cutoff. Then we find

$$V_{\text{eff}}(\phi_c) = \frac{\lambda}{4!}\phi_c^4 + \frac{1}{2}B\phi_c^2 + \frac{1}{4!}C\phi_c^4 + \left(\frac{\lambda\Lambda^2}{64\pi^2}\right)\phi_c^2 + \frac{\lambda^2\phi_c^4}{256\pi^2}\left[\log\left(\frac{\lambda\phi_c^2}{2\Lambda^2}\right) - \frac{1}{2}\right] \ . \tag{5.10}$$

Now we impose the renormalization conditions:

$$\left.\frac{d^2V_{\text{eff}}}{d\phi_c^2}\right|_{\phi_c=0} = 0 \quad\Rightarrow\quad B = -\left(\frac{\lambda\Lambda^2}{32\pi^2}\right) \ , \tag{5.11}$$

$$\left.\frac{d^4V_{\text{eff}}}{d\phi_c^4}\right|_{\phi_c=\mu} = \lambda \quad\Rightarrow\quad C = -\frac{3\lambda^2}{32\pi^2}\left[\log\left(\frac{\lambda\mu^2}{2\Lambda^2}\right) - \frac{11}{3}\right] \ . \tag{5.12}$$

I have chosen to let $\mu = 0$ in the first renormalization condition since that is not IR divergent, but I cannot do that in the definition of λ. As a quick aside, notice that even if we choose the mass to vanish at $\mu = 0$, it will not vanish for another subtraction point; this is the source of the "hierarchy problem" of scalar field theory.

Plugging in these counterterms, we find the final, one loop effective potential to be

$$\boxed{V_{\text{eff}}(\phi_c) = \frac{\lambda}{4!}\phi_c^4 + \frac{\lambda^2\phi_c^4}{256\pi^2}\left[\log\left(\frac{\phi_c^2}{\mu^2}\right) - \frac{25}{6}\right]} \tag{5.13}$$

Notice that there are no longer any $\log(\lambda)$ terms – this is great news, since we would never have found them using ordinary perturbation theory! In addition, the λ counterterm has the general form $C(\mu) = -\beta(\lambda)\log(\mu) + \cdots$ so

$$\beta(\lambda) = +\frac{3\lambda^2}{16\pi^2} \ . \tag{5.14}$$

By keeping track of external momenta, we can also compute Z and hence A – this would give us the anomalous dimension γ. It turns out that at one loop, $\gamma = 0$, which is proved a different way, along with (5.14) in [7]. Finally, notice that with this choice of running, $\frac{dV}{d\mu} = 0$, as it should!

Does this potential have a nontrivial minimum? As a matter of fact it does, at

$$\lambda \log\left(\frac{\langle\phi_c\rangle}{\mu}\right) = -\frac{16\pi^2}{3} + \frac{11}{6}\lambda \ . \tag{5.15}$$

Is this real? Since μ is arbitrary, let us chose $\mu = \langle\phi_c\rangle/2$. This gives us $|\overline{\lambda}(\langle\phi_c\rangle)| \sim 50$ – this is certainly outside the realm of perturbation theory, and so we have no right to take it seriously!

5.3 A Less Trivial Example

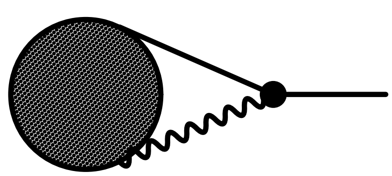
Even though the massless ϕ^4 theory turned out to be something of a dud, there are other theories that are more promising. The simplest example of such a theory is massless scalar electrodynamics

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + |D_\mu\phi|^2 - \frac{\lambda}{6}|\phi|^4 + \text{ counterterms} \ . \tag{5.16}$$

where $\phi \equiv (\phi_1 + i\phi_2)/\sqrt{2}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu \equiv \partial_\mu + ieA_\mu$. It turns out to be very convenient to work in Landau gage, where the photon propagator is

$$-i\frac{g^{\mu\nu} - k^\mu k^\nu/k^2}{k^2 + i\varepsilon} \ ,$$

since then diagrams of the type



vanish with external momenta. This is not true in a general gauge! With these single vertices gone, the only diagrams are the two types in Figure 5.1, where Figure 5.1a have *both* ϕ_1 and ϕ_2 running around the loop. The vertices in Figure 5.1b have a vertex e^2, and it can be shown that the final answer can only depend on $\phi_1^2 + \phi_2^2$, so the final one loop effective potential after renormalization in this case is

$$V_{\text{eff}}(\phi_c) = \frac{\lambda}{6}|\phi_c|^4 + \left(\frac{5\lambda^2}{288\pi^2} + \frac{3e^4}{16\pi^2}\right)|\phi_c|^4\left[\log\left(\frac{\phi_c^2}{\mu^2}\right) - \frac{25}{6}\right] \ . \tag{5.17}$$

Now consider the couplings (e, λ). Even if we try to set $\lambda = 0$, we regenerate it through a one loop diagram in Figure 5.1b with two vertices. This diagram is roughly proportional to $e^4 \log(\Lambda/\mu)$ so NDA tells us that the "natural" size of $\lambda \sim e^4$. So let us *choose* this value for λ – we'll come back to this later. With this choice, we must drop the $\mathcal{O}(\lambda^2)$ terms from (5.17) to be consistent. Let us now search for a minimum of this effective potential. Sure enough, setting the first derivative to zero at $\phi_c = v$ gives

$$\left(\frac{\lambda}{6} - \frac{11e^4}{16\pi^2}\right) v^3 = 0 \ . \tag{5.18}$$

So we find that as long as we chose a very precise value[1] for λ, we *do* have a minimum of the effective potential that breaks the $U(1)$ gauge symmetry! Choosing this value of λ and setting $\mu = v$ gives us

$$\boxed{V_{\text{eff}}(\varphi_c) = \frac{3e^4}{16\pi^2}\varphi_c^4 \left[\log\left(\frac{\varphi_c^2}{v^2}\right) - \frac{1}{2}\right]} \tag{5.19}$$

where I have defined $\varphi_c^2 = \phi_1^2 + \phi_2^2$. Notice that we have removed all dependence of V_{eff} on λ and introduced another parameter v. Thus we have not lost any information (we started with two parameters, we ended with two parameters), but we have replaced a dimensionless parameter with a dimensionful one! This is somewhat surprising, but it is a standard result when doing loops, closely related to the scaling anomaly discussed in the last lecture. This phenomenon has been labeled **dimensional transmutation**.

In this minimum we can also predict the mass of the scalar field, which is no longer zero but it is loop suppressed:

$$m_s^2 \equiv \left.\frac{d^2 V_{\text{eff}}}{d\varphi_c^2}\right|_{\varphi_c = v} = \frac{3e^4}{8\pi^2} v^2 \ . \tag{5.20}$$

The photon mass is given by the usual formula $m_V = ev$, so that both masses are given in terms of the free parameter v. But their ratio is independent of v

$$\frac{m_s^2}{m_V^2} = \frac{3\alpha}{2\pi} \tag{5.21}$$

which is an amusing and generic result. When [14] was published, it was a hopeful possibility that this might describe the Higgs mechanism that breaks the electroweak symmetry. Unfortunately for the authors, this is not the case. But the ideas are still widely applicable.

[1]Notice that this value of λ is consistent with our power counting – this is very important, otherwise it would not work.

You might think we have gone and done something terribly illegal! We chose a very particular value of λ. What if we didn't feel like choosing that? What if we chose a different value? Then we would not satisfy (5.18) except for $v = 0$ and it looks like we can only find a minimum for a very careful tuning of parameters.

The counterargument is that a change of λ can always be compensated for by a change of μ, and since we chose $\mu = v$, this means that moving away from the value of λ defined in (5.18) will ultimately just change v (and e). So the question is: can changes in the running couplings completely capture the effects of moving away from (5.18)? And if so, do we still have a minimum in general? To answer that question, we must study how the renormalization group affects the picture painted in this section.

5.4 "RG Improved" Perturbation Theory

Let us reconsider the ϕ^4 theory we started with, where we calculated the effective potential in (5.13). In fact, you can show that

$$\lim_{\phi_c \to 0} V_{\text{eff}} = -\infty \ , \tag{5.22}$$

and so it looks like $\phi_c = 0$ is a *maximum*, and there is a new minimum at (5.15). The problem with this conclusion is that n loop diagrams generally give factors of $[\lambda \log(\phi_c/\mu)]^n$, so since the minimum has this number $\mathcal{O}(100)$ means that the one loop calculation is not trustworthy, and we had better not take this minimum (or the above maximum) seriously. This realization leads us to ask if there is any way to capture these log effects without actually doing a higher loop diagram, which is quite difficult and a general mess! Fortunately, in many case of interest, there is a way using renormalization group methods, although the precise form of the equations will be slightly different than what you might be used to, since we are no longer working with 1PI coefficients but with effective potentials.

Recall the renormalization group equation:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + \gamma\int d^4x\ \phi_c(x)\frac{\delta}{\delta\phi_c(x)}\right]\Gamma[\phi_c(x)] = 0 \ . \tag{5.23}$$

Doing the momentum expansion of the effective action (5.4) gives us

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}+\gamma\phi_c\frac{\partial}{\partial\phi_c}\right]V_{\rm eff} \;=\; 0\;, \tag{5.24}$$

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}+\gamma\phi_c\frac{\partial}{\partial\phi_c}+2\gamma\right]Z \;=\; 0\;. \tag{5.25}$$

It is useful to work instead with $V^{(4)} \equiv \frac{d^4V}{d\phi_c^4}$ and to define

$$\overline{\beta} \;\equiv\; \frac{\beta}{1-\gamma}\;, \tag{5.26}$$

$$\overline{\gamma} \;\equiv\; \frac{\gamma}{1-\gamma}\;. \tag{5.27}$$

Also notice that μ and ϕ_c can only appear as the ratio μ/ϕ_c since both $V^{(4)}$ and Z are dimensionless. Defining $t \equiv \log(\phi_c/\mu)$, we have

$$\left[-\frac{\partial}{\partial t}+\overline{\beta}\frac{\partial}{\partial\lambda}+4\overline{\gamma}\right]V^{(4)}(t,\lambda) \;=\; 0\;, \tag{5.28}$$

$$\left[-\frac{\partial}{\partial t}+\overline{\beta}\frac{\partial}{\partial\lambda}+2\overline{\gamma}\right]Z(t,\lambda) \;=\; 0\;, \tag{5.29}$$

and the renormalization conditions

$$V^{(4)}(0,\lambda) \;=\; \lambda\;, \tag{5.30}$$

$$Z(0,\lambda) \;=\; 1\;. \tag{5.31}$$

We can solve these equations in the most general case, assuming we know what β and γ are. We get

$$Z(t,\lambda) \;=\; e^{2\int_0^t dt'\;\overline{\gamma}[\overline{\lambda}(t',\lambda)]}\;, \tag{5.32}$$

$$V^{(4)}(t,\lambda) \;=\; \overline{\lambda}(t',\lambda)[Z(t,\lambda)]^2\;, \tag{5.33}$$

where as usual

$$\frac{d\overline{\lambda}}{dt}=\overline{\beta}(\overline{\lambda}),\qquad \overline{\lambda}(0)=\lambda\;. \tag{5.34}$$

Now let us return to the ϕ^4 example. Differentiating (5.13) gives

$$V^{(4)}(t,\lambda)=\lambda+\frac{3\lambda^2 t}{16\pi^2}\;. \tag{5.35}$$

Using (5.14) and $\gamma=0$ we can solve the running equation for $\overline{\lambda}$:

$$\overline{\lambda}(t,\lambda)=\frac{\lambda}{1-3\lambda t/16\pi^2}\;, \tag{5.36}$$

and since $Z = 1$, we have $V^{(4)}(t,\lambda) = \overline{\lambda}(t,\lambda)$. One can now integrate this up to get the total $V_{\text{eff}}(\phi_c)$, using the fact that we already have its form to $\mathcal{O}((\lambda t)^2)$ to fix any constants.

Notice an amazing thing: for our original calculation to be trustworthy, we needed *both* λ and $|\lambda t|$ to be perturbative. But now, we only need $|\overline{\lambda}(t,\lambda)|$ to be perturbative. In particular, as $t \to -\infty$ ($\phi_c \to 0$), $\overline{\lambda} \to 0$ and we are still okay. In fact, V_{eff} is a *minimum* at the origin, and the funky minimum of (5.15) has completely vanished into the Aether! We were right not to trust it.

However, as $t \to \frac{16\pi^2}{3}$, $\overline{\lambda} \to \infty$ and our calculation is garbage. This is called a **Landau Pole**. Its existence tells us that there is a scale at which our approximations break down, and we must give up using our effective field theory.

Now let us repeat this exercise for massless scalar QED, where

$$\Gamma[\phi_c] = \int d^4x[-V_{\text{eff}}(\phi_c) + Z(\phi_c)|D_\mu\phi_c|^2 - \frac{1}{4}H(\phi_c)F_{\mu\nu}F^{\mu\nu} + \cdots] \,. \tag{5.37}$$

We keep our previous renormalization conditions, with $H(\mu) = 1$ as well. Again, we have

$$V^{(4)} = \lambda + \left(\frac{5\lambda^2}{24\pi^2} + \frac{9e^4}{4\pi^2}\right) t \,, \tag{5.38}$$

$$Z = 1 + \frac{3e^2}{8\pi^2} t \,, \tag{5.39}$$

$$H = 1 - \frac{e^2}{24\pi^2} t \,, \tag{5.40}$$

and these all must satisfy

$$\left[-\frac{\partial}{\partial t} + \overline{\beta}\frac{\partial}{\partial\lambda} + \overline{\beta}_e\frac{\partial}{\partial e} + 4\overline{\gamma}\right] V^{(4)}(t,e,\lambda) = 0 \,, \tag{5.41}$$

$$\left[-\frac{\partial}{\partial t} + \overline{\beta}\frac{\partial}{\partial\lambda} + \overline{\beta}_e\frac{\partial}{\partial e} + 2\overline{\gamma}\right] Z(t,e,\lambda) = 0 \,, \tag{5.42}$$

$$\left[-\frac{\partial}{\partial t} + \overline{\beta}\frac{\partial}{\partial\lambda} + \overline{\beta}_e\frac{\partial}{\partial e} + 2\overline{\gamma}\right] H(t,e,\lambda) = 0 \,, \tag{5.43}$$

where $\beta_e = -e\gamma_e$, which just a consequence of the Ward Identity. Solving for these equations at $\phi_c = \mu$ gives

$$\overline{\gamma} = \frac{3e^2}{16\pi^2} \,, \tag{5.44}$$

$$\overline{\beta}_e = \frac{e^3}{48\pi^2} \,, \tag{5.45}$$

$$\overline{\beta} = \frac{1}{4\pi^2}\left(\frac{5}{6}\lambda^2 - 3e^2\lambda + 9e^4\right) \,. \tag{5.46}$$

These equations can be solved, and we get the rather ugly

$$\overline{e}^2(t) = \frac{e^2}{1 - e^2 t/24\pi^2} \; , \tag{5.47}$$

$$\overline{\lambda}(t) = \frac{\overline{e}^2(t)}{10} \left\{ \sqrt{719} \tan\left[\frac{\sqrt{719}}{2} \log \overline{e}^2(t) + \theta \right] + 19 \right\} \; , \tag{5.48}$$

where θ is an integration constant to make sure $\overline{\lambda} = \lambda$ when $\overline{e}^2 = e^2$ $(t = 0)$.

What can we learn from these hideous equations? Let us start by looking at the running of $\overline{e}^2$. Notice that it has a Landau pole, so once again our approximations break down at some high scale (large, positive t). However, for large *negative* t, $\overline{e}^2$ stays small which makes $\log \overline{e}^2$ in the expression for $\overline{\lambda}$ start to grow, and ultimately the tangent will blow up; so we cannot go to the origin, unlike the ϕ^4 case. But *because* of this logarithm, $\overline{\lambda}[\overline{e}^2(t)]$ spans the entire real line, even for a small range of $\overline{e}^2$ – for example, $\overline{e}^2 \in (e^2/2, 2e^2)$. So as long as $\overline{\lambda}$ starts small, we can *always* make it $\mathcal{O}(e^4)$ by adjusting μ, and therefore v, and our previous analysis holds. So indeed, massless scalar QED *does* undergo spontaneous symmetry breaking, and there is no tuning!

5.5 Generalizations of the Coleman-Weinberg Potential

We have seen how computing the effective potential can give us very different qualitative behavior from the naive tree level behavior. But even more generally, we might want to compute the one loop effects to the theory (shifts in masses, couplings, new interactions, etc). The technique developed here will do that for us.

Consider an arbitrary (not necessarily renormalizable) theory of scalars, fermions and gauge fields. We would like to compute the full, one loop potential for the scalars ϕ^a, where I'm indexing the scalars (traditionally called "flavor indices"), although they may or may not fall into a given representation of a symmetry group. Also, I will assume throughout that all scalars are real; the generalization to complex scalars is immediate.

To see how we can put on external legs to a loop, notice that two (and only two) scalars in a given vertex must go into the loop, and the remaining ones are external lines – it is more complicated at higher loops, but for the one loop approximation, this is right. Therefore the vertex is proportional to

$$[M_s^2]^a_b \equiv \frac{\partial^2 V_0}{\partial\phi_a \partial\phi^b} \ , \tag{5.49}$$

where V_0 is the tree level scalar potential. Thus we have n insertions of M_s^2 around the loop and the sum over n, and since it is a loop we must trace over flavor indices. After generalizing the calculation in the text, we find

$$\boxed{\Delta V_s = \frac{c_s}{64\pi^2}\mathrm{Tr}\left\{M_s^4\left[\log\left(\frac{M_s^2}{\Lambda^2}\right) - \frac{1}{2}\right]\right\} + \text{ counterterms}} \tag{5.50}$$

where $c_s = 1, 2$ for real (complex) scalar fields.

When computing fermion loops, we can write the most general vertex

$$\overline{\psi}_a [M_F]^{ab} \psi_b + \text{ 4 or more fermions,}$$

where here I am using Dirac fermions and I am ignoring any more than two fermions, since such operators will not contribute to the one loop scalar potential. Here M_F is a general matrix that can depend on the scalars. Loops will contribute chains of the form

$$\cdots M_F \frac{1}{\not{q}} M_F \frac{1}{\not{q}} \cdots = \cdots (M_F M_F^\dagger)\frac{1}{q^2}\cdots \tag{5.51}$$

This can be seen by including Dirac indices on $M_F^{ab} \equiv A^{ab} + i\gamma^5 B^{ab}$. There are some important differences with the scalar case:

1. Since fermions have arrows, the $\frac{1}{2}$ in (5.8) is missing since reflections are different graphs.

2. Odd terms vanish when tracing over γ matrices, and this compensates for the $\frac{1}{2}$ we dropped from (1).

3. Fermi-Dirac statistics means that there is an overall minus sign in the potential.

4. Tracing over spinor indices gives a factor of 2 for each Weyl degree of freedom.

So in conclusion, we have

$$\boxed{\Delta V_f = -\frac{c_f}{64\pi^2}\mathrm{Tr}\left\{(M_F M_F^\dagger)^2\left[\log\left(\frac{(M_F M_F^\dagger)}{\Lambda^2}\right) - \frac{1}{2}\right]\right\} + \text{ counterterms}} \tag{5.52}$$

where $c_f = 1, 2$ for Weyl (Dirac) fields.

Finally, we can do a more general analysis of (possibly nonabelian) gauge fields, which couple to the scalars like

$$\frac{1}{2} g^{\mu\nu} A^a_\mu A^b_\nu [M_V^2]^{ab} \ ,$$

where

$$[M_V^2]^{ab} \equiv g_a g_b (T^a \phi) \cdot (T^b \phi) \qquad \text{no sum over } a, b \ .$$

This is the only relevant interaction in Landau gauge. Now it it identical to the scalar case, except there is an overall

$$\mathrm{Tr}[g^{\mu\nu} - k^\mu k^\nu / k^2] = g_{\mu\nu} [g^{\mu\nu} - k^\mu k^\nu / k^2] = 3 \ .$$

Therefore we have

$$\boxed{\Delta V_v = \frac{3}{64\pi^2} \mathrm{Tr} \left\{ M_V^4 \left[\log \left(\frac{M_V^2}{\Lambda^2} \right) - \frac{1}{2} \right] \right\} + \text{ counterterms}} \tag{5.53}$$

The full one loop effective potential is then just the sum of (5.50), (5.52), and (5.53).

Chapter 6

Gauge Symmetries

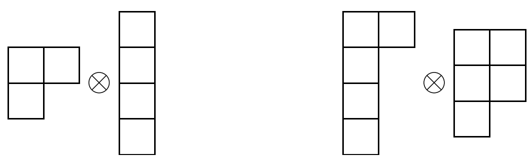
I wanted to have a series of lectures on gauge symmetries, including BRST invariance, R_ξ gauges and the relationship of gauge symmetry to renormalizability, but I had no time. Also, although interesting, I have never seen this stuff used in practical applications, only in more formal developments. So I'm leaving this chapter blank for the moment, and perhaps I'll come back to it in the future...

Advanced Topics in EFT

Homework 1 Due: Monday, October 6, 5PM

1. (a) Compute the dimensions of the following representations in $SU(N)$:

(b) Compute the following products of representations in $SU(5)$:



Express your final answers in terms of the dimensions, being sure to include whether or not it's the conjugate representation, if necessary.

2. In class we showed that in $SU(3)$: $\mathbf{8} \otimes \bar{\mathbf{3}} = \overline{\mathbf{15}} \oplus \mathbf{6} \oplus \bar{\mathbf{3}}$. Prove this explicitly by symmetrizing indices. *Hint:* Recall that invariant tensors are either symmetric and traceless, or antisymmetric. Be sure to keep track of covariant vs contravariant indices – it will make your life much easier!

3. Consider the quantum D=3 simple harmonic oscillator.

(a) When written in terms of creation/annihilation operators, it is immediately clear that the system has an $SU(3)$ symmetry. Find the conserved charges of this symmetry.

(b) Since this is a symmetry of the Hamiltonian, we know that there will be degeneracies among the energy eigenstates. Find these degeneracies and show that they completely explain the degeneracies that you calculated in your quantum class! Classify the energy eigenstates in terms of SU(3) representations.

(c) Show that built into this symmetry, there lives an $SU(2)$ subgroup that can be identified with the overall rotational symmetry of the problem. Use this to derive the angular momentum operator $\mathbf{L}$.

(d) Suppose I add a term to the Hamiltonian $\Delta H = \alpha \mathbf{L}^2$. Describe what happens to the first four energy levels.

Advanced Topics in EFT

Homework 2 Due: Monday, October 20, 5PM

1. The Cartan Algebra of the group $SU(n+1)$ in the fundamental representation is spanned by the traceless matrices

$$H_k = \frac{1}{\sqrt{2k(k+1)}}\text{Diag}\{1,\ldots,1,-k,0,\ldots,0\} \qquad k=1,\ldots,n \tag{.1}$$

where there are k "1"s and we're following the usual convention that $\text{Tr}[T^aT^b] = \frac{1}{2}\delta^{ab}$. The other generators are various embeddings of the Pauli matrices; see your favorite Group theory text for details. Consider the symmetry breaking $SU(4) \to SU(3)\times U(1)$.

   (a) Consider how a field in the fundamental of $SU(4)$ transforms under the symmetry after breaking. Find both its representation under SU(3) and its charge under the $U(1)$. *Hint:* A fundamental of $SU(N)$ transforms as $\delta^a F^i = i[T^a]^i_j F^j$.

   (b) Repeat the above for a field in the adjoint of $SU(4)$. *Hint:* An adjoint of $SU(N)$ transforms as $\delta^a A^b = i[T^a, A^b]$.

   (c) Assume that there is a term in the Hamiltonian of the system that transforms in the fundamental (**4**) representation. Using the generalization of the Wigner-Eckart Theorem to $SU(N)$, how many reduced matrix elements are required to describe the transitions $(1,0,0) \to (3,0,0), (1,1,0)$?

2. Using arguments of effective field theory, explain why the sky is blue:

   (a) Identify the relevant degrees of freedom and energy/length scales.

   (b) Write down the free action and construct a scaling rule for each relevant field.

   (c) Find the leading operators and use them to "compute" the scattering cross section of light in the atmosphere.

Note: this is *not* a Jackson problem – do not treat it as such!!

3. Finish what we started in class by embedding QED into the $SU(3)$ Chiral Lagrangian.

   Find the currents in terms of the Σ field.

Advanced Topics in EFT

Homework 3 Due: Thursday, November 6, 12 PM

1. Reading Homework: Read and understand K. Fujikawa, *Phys. Rev. Lett.* **42**, 1195 (1979).

2. Consider an $SU(N)$ gauge theory with L left handed fermions and R right handed fermions. Suppose that the L fermions transform under the $\square\!\square$ representation of the gauge group, while the R fermions transform as antifundamentals. For what values (if any) of (L, R) does this theory make sense?

3. Consider a very generic situation - an $SU(N)$ gauge symmetry ($N \geq 3$) with an $SU(F)_L \times SU(F)_R$ global flavor symmetry, where F left handed fermions[1] transform in the fundamental of $SU(F)_L$ and a singlet of $SU(F)_R$; and F right handed fermions transform in the antifundamental of $SU(F)_R$ and a singlet of $SU(F)_L$.

   (a) Derive a condition that a $U(1)$ must satisfy so that it is not anomalous if both L and R fermions are fundamentals (antifundamentals) under the gauge group.

   (b) Now add a new fermion that transforms in the adjoint of the gauge symmetry, but a singlet under the flavor symmetry. What charge must it have to keep the symmetries non-anomalous?

[1] Recall that ALL fermions are in the $(\frac{1}{2}, 0)$ representation of the Lorentz group, so "right handed fermions" means "left handed *anti*fermions".

Advanced Topics in EFT

Homework 4 Due: Friday, December 5, 5PM

1. Read, understand, absorb S. Coleman, E. Weinberg, *Phys. Rev.* **D7** 1888 (1973).

2. Consider a massless real scalar and a massless Weyl fermion with interactions:

$$\mathcal{L}_I = g\phi(\psi\psi + \bar{\psi}\bar{\psi}) - \frac{\lambda}{4!}\phi^4 \tag{.1}$$

   (a) Compute the 1 loop Coleman-Weinberg effective potential for the scalar.

   (b) Compute the beta functions and anomalous dimensions, and use them to solve for the running couplings. As above, use MS scheme.

   (c) Construct the RG-improved 1 loop effective potential.

3. Consider supersymmetric QED: there is a (Dirac) electron (e_L, e_R); *two* complex scalar fields (call them $\tilde{e}_L, \tilde{e}_R$) with the same quantum numbers and mass as the electron (L) and positron (R); a photon; and a neutral (Weyl) fermion (the gaugino λ) that couples to the electrons and scalars with coupling e, the gauge charge:

$$\begin{aligned}\mathcal{L}_0 \;=\;& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\tilde{e}_L|^2 + |D_\mu\tilde{e}_R|^2 - m^2(|\tilde{e}_L|^2 + |\tilde{e}_R|^2)\\ & + \bar{e}_L i\bar{\sigma}\cdot D e_L + e_R i\sigma\cdot D\bar{e}_R - m(e_L e_R + \bar{e}_L\bar{e}_R)\\ & + \bar{\lambda} i\bar{\sigma}\cdot\partial\lambda + e\sqrt{2}[\lambda e_L\tilde{e}_L^* - \lambda e_R\tilde{e}_R^*] + h.c.\end{aligned} \tag{.2}$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the usual covariant derivative. Now add a single, massless, neutral (real) scalar field ϕ.

(a) Write down the most general, renormalizable action (in $d = 4$) describing the couplings of ϕ.

(b) Construct the one loop Coleman-Weinberg effective potential for the scalars. As always, use Landau gauge and minimal subtraction (that is – treat Λ as your subtraction scale).

(c) Using this potential, compute the mass of the scalars. Explain your results.

Bibliography

[1] A. E. Blechman, *A Mathematics Primer for Physics Graduate Students*, . Updated regularly; see webpage for a copy.

[2] H. Georgi, *Lie Algebras in Particle Physics: From Isospin to Unified Theories, 2nd Edition*, *Front. Phys.* **54** (1999). Published by Westview Press ABP, Perseus Books Group.

[3] R. N. Cahn, *Semisimple Lie Algebras and Their Representations*, *Front. Phys.* **59** (1984). Available online: http://phyweb.lbl.gov/∼rncahn/www/liealgebras/book.html.

[4] R. Slansky, *Group Theory for Unified Model Building*, *Phys. Rept.* **79** (1981) 1–128.

[5] S. R. Coleman, J. Wess, and B. Zumino, *Structure of Phenomenological Lagrangians. 1*, *Phys. Rev.* **177** (1969) 2239–2247.

[6] C. G. Callan, S. R. Coleman, J. Wess, and B. Zumino, *Structure of Phenomenological Lagrangians. 2*, *Phys. Rev.* **177** (1969) 2247–2250.

[7] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, .

[8] H. Georgi, *Weak Interactions and Modern Particle Theory*, . Menlo Park, USA: Benjamin/Cummings (1984). Available online: http://www.people.fas.harvard.edu/∼hgeorgi/weak.pdf.

[9] H. K. Dreiner, H. E. Haber, and S. P. Martin, *Two-component Spinor Techniques and Feynman Rules for Quantum Field Theory and Supersymmetry*, 0812.1594.

[10] A. V. Manohar and M. B. Wise, *Heavy Quark Physics*, *Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol.* **10** (2000) 1–191.

[11] K. Fujikawa, *Path Integral Measure for Gauge Invariant Fermion Theories*, *Phys. Rev. Lett.* **42** (1979) 1195.

[12] S. Coleman, *Aspects of symmetry*, . Cambridge, UK: Cambridge University Press.

[13] P. H. Ginsparg, *APPLIED CONFORMAL FIELD THEORY*, `hep-th/9108028`.

[14] S. R. Coleman and E. J. Weinberg, *Radiative Corrections as the Origin of Spontaneous Symmetry Breaking*, *Phys. Rev.* **D7** (1973) 1888–1910.