Introduction to Supersymmetry and Supergravity

or Supersymmetry in 12954 minutes

R. Boels and F. Brümmer
Universität Hamburg, Winter term 2012/13

Last updated April 12, 2013

Contents

Chapter 1

Global supersymmetry: Foundations

1.1 Introduction and motivation

1.1.1 Introductory remarks

Supersymmetry is a symmetry which transforms fermions into bosons and vice versa. These lecture notes deal with supersymmetric quantum field theories, which are interesting to study for several reasons. Here are some of them:

- Supersymmetry is appealing from the mathematical point of view as an extension of Poincaré space-time symmetry, and a surprisingly unique one as it happens. In physicist's terms, in Nature we observe fields which transform as scalars, spinors or vectors under the Lorentz algebra. We also observe conserved charges which transform as Lorentz scalars (such as gauge charges) or as a Lorentz vector (energy-momentum). Conserved charges are associated with continuous symmetries, by Noether's theorem: gauge charges with gauge symmetries, and energy-momentum with translational symmetry. It seems natural to ask if a conserved charge can also transform as a spinor, and what kind of symmetry a conserved spinorial charge would generate. The answer turns out to be supersymmetry.
- Supersymmetry may be relevant for high-energy particle physics. If the world were supersymmetric, the quantum field theory describing elementary particle interactions would be much better behaved in the ultraviolet than otherwise. Supersymmetry also facilitates the unification of the Standard Model gauge interactions into a single force, and it predicts new particles which could naturally constitute dark matter. The particles predicted by supersymmetry are currently being searched for at the LHC collider.
- Supersymmetric models are more constrained, and therefore often easier to understand, than non-supersymmetric ones. As often in physics, the more symmetric a system is, the simpler it becomes, and the more likely it is that it can be treated analytically. Supersymmetry, in particular, can aid in understanding non-perturbative

effects in gauge theory; it sometimes allows to obtain exact results in systems whose non-supersymmetric cousins are completely incalculable because they are strongly coupled. For instance, supersymmetric QCD can be used as a toy models to study effects which have been proposed to govern the low-energy behaviour of ordinary QCD. Maximally supersymmetric gauge theory in four dimensions (so-called $\mathcal{N} = 4$ supersymmetric Yang-Mills theory) is currently a subject of intense research as a prototype gauge theory.[1]

- Supersymmetry might help to understand the elusive theory of quantum gravity. In fact, string theory as a candidate theory of quantum gravity (or of everything) appears to rely on supersymmetry for the stability of its vacuum. Its low-energy effective field theories are therefore also naturally supersymmetric. Among the most powerful applications of superstring theory are concrete realizations of the holographic principle, relating theories of quantum gravity to non-gravitational gauge theories. In in its best-understood manifestation, the AdS/CFT correspondence, it relates type IIB superstring theory to four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Supersymmetry plays a crucial role in establishing and exploiting the correspondence between these two theories.

The plan for these lecture notes is to touch upon all these aspects of supersymmetry, up to and including an introduction to supergravity, which is a supersymmetric version of General Relativity (or equivalently a local version of supersymmetry). They are organized as follows: We start with a chapter on the foundations of global $\mathcal{N} = 1$ supersymmetric field theory in four dimensions. We introduce the supersymmetry algebra as an extension of the Poincaré algebra and develop its representation theory. The concept of superfields and superspace is introduced, and put to use in writing down supersymmetric Lagrangians. We introduce supersymmetric gauge theories and general supersymmetric σ-models, and discuss the fundamentals of supersymmetry breaking. The second chapter consists mostly of applications and examples. We also give an introduction to non-perturbative methods in supersymmetric QCD, and their role in establishing exact analytic results in a domain where the theory is strongly coupled. We also introduce the supersymmetric Standard Model as a possible theory for new physics at the TeV scale, and discuss its main phenomenological features. Furthermore we discuss some selected topics in extended supersymmetry, especially $\mathcal{N} = 4$. The third chapter deals with local supersymmetry or supergravity. We finish with an outlook towards more advanced topics.

1.1.2 Literature

Lots of books, review papers, lecture notes and research papers have been written about SUSY. The following list contains a number of books and reviews which the authors found useful (and from which, on occasion, they shamelessly stole their material). Particularly recommended items are marked with a star ★. Ref. [1] is the standard reference for the technicalities of $\mathcal{N} = 1$ SUSY in four dimensions, whose conventions we largely follow in the present notes. Refs. [2–7] are general textbooks. Refs. [8–12] are introductory or general reviews or lecture notes, most of which are freely available. Refs. [13, 14] are textbooks centered on the application of SUSY to TeV-scale particle physics and collider phenomenology, although they also contain general introductions to the subject.

[1] It has even half-jokingly been called the "harmonic oscillator of the 21st century" or "the hydrogen atom of four dimensional field theories" (both N. Arkani-Hamed, public communication).

Refs. [15–17] are reviews mostly concerned with SUSY phenomenology, of which Ref. [17] is the most up-to-date one (and also happens to be freely available). Non-perturbative effects in supersymmetric QCD are treated in Refs. [18–23].

[1] J. Wess and J. Bagger, "Supersymmetry and supergravity," Princeton, USA: Univ. Pr. (1992) 259 p ★

[2] P. C. West, "Introduction To Supersymmetry And Supergravity," Singapore, Singapore: World Scientific (1986) 289p

[3] D. Bailin and A. Love, "Supersymmetric gauge field theory and string theory," Bristol, UK: IOP (1994) 322 p. (Graduate student series in physics)

[4] I. L. Buchbinder and S. M. Kuzenko, "Ideas and methods of supersymmetry and supergravity: Or a walk through superspace," Bristol, UK: IOP (1998) 656 p

[5] S. Weinberg, "The quantum theory of fields. Vol. 3: Supersymmetry," Cambridge, UK: Univ. Pr. (2000) 419 p

[6] P. Binétruy, "Supersymmetry: Theory, experiment and cosmology," Oxford, UK: Oxford Univ. Pr. (2006) 520 p

[7] D. Z. Freedman and A. Van Proeyen, "Supergravity," Cambridge, UK: Cambridge Univ. Pr. (2012) 607 p ★

[8] S. J. Gates, M. T. Grisaru, M. Rocek and W. Siegel, "Superspace Or One Thousand and One Lessons in Supersymmetry," Front. Phys. **58** (1983) 1 [hep-th/0108200]

[9] M. F. Sohnius, "Introducing Supersymmetry," Phys. Rept. **128** (1985) 39.

[10] N. Dragon, U. Ellwanger and M. G. Schmidt, "Supersymmetry And Supergravity," Prog. Part. Nucl. Phys. **18** (1987) 1.

[11] M. Drees, "An Introduction to supersymmetry," hep-ph/9611409.

[12] A. Bilal, "Introduction to supersymmetry," hep-th/0101055.

[13] M. Drees, R. Godbole and P. Roy, "Theory and phenomenology of sparticles: An account of four-dimensional N=1 supersymmetry in high energy physics," Hackensack, USA: World Scientific (2004) 555 p

[14] H. Baer and X. Tata, "Weak scale supersymmetry: From superfields to scattering events," Cambridge, UK: Univ. Pr. (2006) 537 p

[15] H. P. Nilles, "Supersymmetry, Supergravity and Particle Physics," Phys. Rept. **110** (1984) 1.

[16] H. E. Haber and G. L. Kane, "The Search for Supersymmetry: Probing Physics Beyond the Standard Model," Phys. Rept. **117** (1985) 75.

[17] S. P. Martin, "A Supersymmetry primer," In *Kane, G.L. (ed.): Perspectives on supersymmetry II* 1-153 [hep-ph/9709356]. ★

[18] J. Terning, "Modern supersymmetry: Dynamics and duality," (International series of monographs on physics. 132) ★

[19] J. Terning, "TASI 2002 lectures: Nonperturbative supersymmetry," hep-th/0306119.

[20] K. A. Intriligator and N. Seiberg, "Lectures on supersymmetric gauge theories

and electric - magnetic duality," Nucl. Phys. Proc. Suppl. **45BC** (1996) 1 [hep-th/9509066].

[21] M. E. Peskin, "Duality in supersymmetric Yang-Mills theory," hep-th/9702094.

[22] M. A. Shifman and A. I. Vainshtein, "Instantons versus supersymmetry: Fifteen years later," In *Shifman, M.A.: ITEP lectures on particle physics and field theory, vol. 2* 485-647 [hep-th/9902018].

[23] A. Wipf, "Non-perturbative methods in supersymmetric theories," hep-th/0504180.

1.1.3 On conventions

One eternally confusing aspect of supersymmetry is that its study necessarily involves picking a slew of conventions for signs, factors of two and so on. These conventions can change from paper to paper and from book to book (and sometimes, from equation to equation, or even from left-hand side to right-hand side. . .) We try to stick to the book by Wess and Bagger [1] as the main source for our conventions not because we particularly like it, but since this has proven to be at least internally consistent. There are almost certainly convention errors in this text and in the lecture. Please let us know if you find any so we can correct.

1.2 Space-time symmetries in relativistic field theory

1.2.1 The Lorentz and Poincaré algebras

If a symmetry is to relate particles of different spin, it must involve the particles' space-time transformation properties. We therefore begin by reviewing some basics about space-time symmetries. Since this symmetry is built into any theory of nature, we will spend some time on this.

For now we work on flat Minkowski space-time (we will deal with curved space-times later in the course when we come to supergravity). We use the "mostly plus" signature of the metric, $\eta = \mathrm{diag}(-1, 1, 1, 1)$. The metric is used to raise and lower indices,

$$x^\mu = \eta^{\mu\nu} x_\nu \,, \qquad x_\mu = \eta_{\mu\nu} x^\nu \,. \tag{1.1}$$

The isometries of Minkowski space consist of Lorentz transformations and translations. They act on the coordinates as

$$x^\mu \;\rightarrow\; \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{1.2}$$

where Λ is an $\mathrm{O}(1,3)$ matrix parameterizing a Lorentz transformation, and a is a vector parameterizing a translation. An $\mathrm{O}(1,3)$ matrix satisfies $\eta \Lambda^T \eta = \Lambda$, so leaves the metric invariant.

Among the Lorentz transformations, *proper orthochronous Lorentz transformations* are particularly interesting because they are continuously connected to the identity: Denoting by $\mathrm{SO}(1,3)^\uparrow$ the subgroup of $\mathrm{O}(1,3)$ matrices with determinant +1 ("proper") and $\Lambda^0{}_0 > 0$ ("orthochronous"), we can always write any Lorentz transformation $\Lambda \in \mathrm{SO}(1,3)^\uparrow$ as an exponentiated "infinitesimal" transformation:

$$\Lambda^{\mu}{}_{\nu} = \exp\left(-\frac{i}{2}\omega_{\kappa\lambda}(M^{\kappa\lambda})^{\mu}{}_{\nu}\right) . \tag{1.3}$$

Here $\omega_{\kappa\lambda}$ is a suitable real antisymmetric matrix which contains the rotation angles and boost parameters, and the $M^{\kappa\lambda}$ are 4×4 matrices given by [2]

$$(M^{\kappa\lambda})_{\mu\nu} = i\left(\delta^{\kappa}{}_{\mu}\delta^{\lambda}{}_{\nu} - \delta^{\kappa}{}_{\nu}\delta^{\lambda}{}_{\mu}\right) . \tag{1.4}$$

The matrices M satisfy the *Lorentz algebra*

$$\boxed{[M^{\kappa\lambda}, M^{\rho\sigma}] = i\left(\eta^{\lambda\rho}M^{\kappa\sigma} - \eta^{\kappa\rho}M^{\lambda\sigma} - \eta^{\lambda\sigma}M^{\kappa\rho} + \eta^{\kappa\sigma}M^{\lambda\rho}\right) .} \tag{1.5}$$

More generally, any set of six $n \times n$ matrices arranged in an antisymmetric tensor $\widetilde{M}^{\kappa\lambda}$ and satisfying the commutation relations of Eq. (1.5) (with M replaced by $\widetilde{M}$) defines an n-dimensional *representation* of the Lorentz algebra. The exponentiated matrices

$$\widetilde{\Lambda} = \exp\left(-\frac{i}{2}\omega_{\kappa\lambda}\widetilde{M}^{\kappa\lambda}\right) \tag{1.6}$$

form the corresponding representation of the proper ortochronous Lorentz group. The matrices in equation (1.4) generate what is called the *vector representation* or *defining representation*.

Examples:

- A trivial example is the one-dimensional representation $\widetilde{M}^{\kappa\lambda} = 0$, which generates only the identity transformation.
- A familiar nontrivial example, besides the defining representation above, is the four-dimensional *Dirac representation* with

$$\widetilde{M}^{\kappa\lambda} = \gamma^{\kappa\lambda} \equiv \frac{i}{4}[\gamma^{\kappa}, \gamma^{\lambda}] \tag{1.7}$$

  where the γ^{μ} are 4×4 matrices satisfying the *Clifford algebra*

$$\{\gamma^{\mu}, \gamma^{\nu}\}^{\alpha}{}_{\beta} = 2\eta^{\mu\nu}\delta^{\alpha}{}_{\beta} . \tag{1.8}$$

In a Lorentz-invariant field theory, the fields should transform *covariantly*: Each field $\phi(x)$ should take its values in some n-dimensional vector space which carries a representation of the Lorentz algebra, such that a proper orthochronous Lorentz transformation acts as

$$\boxed{\phi(x) \;\rightarrow\; \widetilde{\Lambda}\,\phi(\Lambda^{-1}x) .} \tag{1.9}$$

[2]Notice that the matrices $(M^{\kappa\lambda})_{\mu\nu}$, with two indices downstairs as defined by Eq. (1.4), are hermitian. However, the matrices $(M^{\kappa\lambda})^{\mu}{}_{\nu}$ with one index raised as they appear in the exponential Eq. (1.3), are not: The Lorentz group as a non-compact group has no finite-dimensional unitary representations (which would be generated by hermitian matrices).

Examples:

- A vector field A^μ transforms according to the vector representation which was defined in Eq. (1.4):
$$A^\mu(x) \to \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) \tag{1.10}$$
- A scalar transforms trivially: $\phi(x) \to \phi(\Lambda^{-1}x)$.
- A Dirac spinor ψ has four components ψ^α which transform according to
$$\psi^\alpha(x) \to \widetilde{\Lambda}^\alpha{}_\beta \psi^\beta(\Lambda^{-1}x)\,, \tag{1.11}$$
where $\widetilde{\Lambda}$ is constructed as in Eq. (1.6) with $\widetilde{M}$ as in Eq. (1.7).

A representation is called *reducible* if the space it acts on can be decomposed into a direct sum of subspaces which do not mix under symmetry transformations. It is called *irreducible* otherwise. The trivial and vector representations of the Lorentz algebra are irreducible, while the Dirac representation is reducible (as we will see later). Any representation can be decomposed into a direct sum of irreducible representations, with the transformation matrices taking a block-diagonal form in a suitable basis.

A useful statement about irreducible representations is *Schur's lemma*: Let V be a vector space carrying some representation r of a Lie algebra, and A be a linear map $A:\ V \to V$ which commutes with all generators. Then A is constant on each subspace of V on which r acts irreducibly.

Given representations r and r', acting on vector spaces V_r and $V_{r'}$, we can always construct the *product representation* $r\otimes r'$ which acts on the tensor product space $V\otimes V'$. For instance, if $\phi \in V_r$ transforms as $\phi \to \Lambda_r\phi$ and $\phi' \in V_{r'}$ transforms as $\phi' \to \Lambda_{r'}\phi'$, then the product representation acts as $\phi\otimes\phi' \to (\Lambda_r\phi)\otimes(\Lambda_{r'}\phi')$, and on the entire product space $V_r \otimes V_{r'}$ by extension. Put more simply, attaching any number of representation indices to an object will make it transform covariantly.

Example:

- If r is the defining vector representation of the Lorentz algebra, then an n-index space-time tensor $T^{\mu_1\cdots\mu_n}$ transforms under the (generally reducible) representation $r^{\otimes n} \equiv r\otimes r\otimes\ldots\otimes r$ (n factors). That is,
$$T^{\mu_1\cdots\mu_n} \to \Lambda^{\mu_1}{}_{\nu_1}\cdots\Lambda^{\mu_n}{}_{\nu_n} T^{\nu_1\cdots\nu_n}\,. \tag{1.12}$$

The decomposition of a product into a direct sum of irreducible representations is called *Clebsch-Gordan decomposition.*

To classify all irreducible representations of the Lorentz algebra, it is instructive to write the generators and commutation relations of Eq. (1.5) in a non-covariant basis. The generators of rotations and Lorentz boosts are
$$L^i = \frac{1}{2}\epsilon^{ijk}M_{jk}, \qquad K^i = M^{0i} \qquad (i,j,k = 1\ldots 3)\,. \tag{1.13}$$
and we define
$$\boldsymbol{J}_\pm = \frac{1}{2}\left(\boldsymbol{L} \pm i\boldsymbol{K}\right)\,. \tag{1.14}$$

(j_+, j_-)	name	dimension
$(0,0)$	scalar	1
$(\frac{1}{2},0)$	left-handed Weyl spinor	2
$(0,\frac{1}{2})$	right-handed Weyl spinor	2
$(1,0)$	(imaginary) self-dual 2-form	3
$(0,1)$	(imaginary) anti-self-dual 2-form	3
$(\frac{1}{2},\frac{1}{2})$	vector (gauge field)	4
$(\frac{1}{2},1)$	left-handed Rarita-Schwinger field (gravitino)	6
$(1,\frac{1}{2})$	right-handed Rarita-Schwinger field (gravitino)	6
$(1,1)$	traceless symmetric 2-index tensor (graviton)	9

Table 1.1: Some irreducible representations of the Lorentz algebra. The Weyl spinor and vector representation will be treated in more detail in Section 1.2.2. The $(1,0)$ and $(0,1)$ representations are investigated in the Exercises. The gravitino and graviton representations will be dealt with when we turn to supergravity later in these Lectures.

It can be shown (→ Exercises) that the commutation relations Eq. (1.5) become

$$[J^i_\pm, J^j_\pm] = i\,\epsilon^{ijk} J^k_\pm\,, \qquad [J^i_\pm, J^j_\mp] = 0\,. \tag{1.15}$$

The Lorentz algebra thus splits into two copies of the angular momentum algebra su(2), whose representation theory is familiar from quantum mechanics. (More precisely, it is isomorphic to the complex Lie algebra $\mathrm{sl}(2,\mathbb{C})$, which is the complexification of the real algebra su(2).) Thus

> each irreducible $\mathrm{so}(1,3)$ representation is characterized by a pair of half-integer "spins" (j_+, j_-)

where j_+ corresponds to the $\boldsymbol{J}_+$ and j_- to the $\boldsymbol{J}_-$ algebra, ${\boldsymbol{J}_\pm}^2 = j_\pm(j_\pm+1)\mathbb{1}$. To decompose a product representation $(j_+, j_-) \otimes (j'_+, j'_-)$ into irreducible representations, one can then simply apply the well-known composition rules for angular momentum to the $\boldsymbol{J}_+$ and to the $\boldsymbol{J}_-$ algebra separately. Table 1.1 lists some irreducible representations which will be relevant for us. As a basic check, observe that the dimensions listed in the third column match the dimensions of the su(2) representations in the first column.

Translations are easily incorporated by supplementing the Lorentz algebra with four more generators P^μ, leading to the *Poincaré algebra* defined by Eq. (1.5) and by

$$\boxed{\begin{aligned} &[P^\mu, P^\nu] = 0, \\ &[M^{\kappa\lambda}, P^\mu] = i\left(\eta^{\kappa\mu} P^\lambda - \eta^{\mu\lambda} P^\kappa\right)\,. \end{aligned}} \tag{1.16}$$

The classification of positive-energy unitary irreducible representations ("particles" for short) of the Poincaré algebra is a famous classic result by Wigner. Let us outline the Wigner classification, leaving some gaps to be filled in the exercises. We start by defining the *Pauli-Lubanski vector* W by

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\kappa\lambda} M^{\nu\kappa} P^\lambda \qquad\qquad (\epsilon_{0123} = -1)\,. \tag{1.17}$$

The operators

$$C_1 \equiv -P^\mu P_\mu\,, \qquad C_2 \equiv W^\mu W_\mu \tag{1.18}$$

commute with all Poincaré generators (→ Exercises): they are *Casimir operators*, taking constant values on each irreducible representation by Schur's lemma. The eigenvalues of the Casimir operators will be used to characterize a particle. The eigenvalue m^2 of C_1 is called the *mass*(-squared); we will restrict to particles with $m^2 \geq 0$. A further technical condition is to restrict to particles with $\langle\Psi| P^0 |\Psi\rangle > 0$ ("positive energy"). There are then two cases to be distinguished:

1. $m^2 > 0$. In this case we can boost to the rest frame of the particle where $P^\mu = (m, 0, 0, 0)$, such that the components of W take the form

$$W_0 = 0, \quad \boldsymbol{W} = m\,\boldsymbol{S}, \qquad S_i = \frac{1}{2}\epsilon_{ijk}M^{jk} \tag{1.19}$$

   with

$$[S_i, S_j] = i\,\epsilon_{ijk}S_k\,. \tag{1.20}$$

   The S_i generate the *little group* of Lorentz transformations which leave the rest frame invariant – the rotation group in three dimensions. We have $\boldsymbol{S}^2 = s(s+1)\mathbb{1}$ on each irreducible representation for some $s \in \frac{1}{2}\mathbb{N}$, and therefore

$$C_2 = m^2\,s(s+1)\mathbb{1} \tag{1.21}$$

   with s called the *spin*.

2. $m^2 = 0$. In the massless case we can choose the lightcone frame $P^\mu = (E, 0, 0, E)$. In this frame frame the Pauli-Lubanski vector is

$$W_0 = E\,L, \quad W_3 = E\,L, \quad W_1 = E\,T_1, \quad W_2 = E\,T_2\,. \tag{1.22}$$

   where

$$T_1 \equiv M_{23} + M_{02}, \quad T_2 \equiv M_{13} + M_{01}, \quad L \equiv M_{12}\,. \tag{1.23}$$

   The operators T_1, T_2 and L generate the group ISO(2) of translations and rotations in two-dimensional Euclidean space. Indeed they satisfy the commutation relations

$$[T_1, T_2] = 0, \quad [L, T_1] = i\,T_2, \quad [L, T_2] = -i\,T_1\,. \tag{1.24}$$

   The Casimir operator of ISO(2) is $T_1^2 + T_2^2$, which is semi-positive definite; thus on any irreducible representation

$$T_1^2 + T_2^2 = \mu^2\,\mathbb{1}, \qquad \mu^2 \geq 0\,. \tag{1.25}$$

   If $\mu^2 > 0$, the representation turns out to be unphysical. Within quantum field theory, there is no known fundamental reason why such particles should not exist (although it is difficult to construct self-consistent theories in which they are coupled to ordinary particles), but they do not seem to be realized in nature. By contrast, if $\mu = 0$ then T_1 and T_2 act trivially, and the group action is generated by the single abelian rotation generator L. The representation space is therefore one-dimensional, with the single non-trivial state $|\lambda\rangle$ satisfying

$$L\,|\lambda\rangle = \lambda\,|\lambda\rangle\ . \tag{1.26}$$

   What values can λ take? Operators $e^{i\varphi L}$ describe spatial rotations by an angle φ along the direction of particle motion. For $\varphi = 2\pi$ this should amount to a phase factor of $+1$ or -1, and the condition $e^{2\pi i\lambda} = \pm 1$ restricts λ to be half-integral. λ is called the particle's *helicity*.

Let us summarize:

> The behaviour of a massive particle under continuous space-time transformations is captured by two quantum numbers, its mass m and its spin $s \in \frac{1}{2}\mathbb{N}$. A massless particle is characterized by a single quantum number, its helicity $\lambda \in \frac{1}{2}\mathbb{Z}$.

Note that under parity transformations $(x^0, \boldsymbol{x}) \rightarrow (x^0, -\boldsymbol{x})$ the helicity changes sign. A CPT invariant theory whose spectrum contains a state of helicity $+\lambda$ must therefore also contain a state with helicity $-\lambda$. For instance, a photon has two polarizations (with helicities ± 1), the same is true for a graviton (with helicities ± 2), and a left-handed Weyl spinor always comes with its right-handed conjugate.

From the general classification follow the properties of specific states within a representation. This usually requires some choices, as a moment thought on spin shows. For spin, we have to specify apart from the total spin also a spin along a specific axis. The states are then a function of two variables, usually called s for the total spin and s_z for the eigenvalue of spin along an axis. In the case at hand this translates to a choice of momentum and, for massive particles only, a choice of spin axis. In particular, states are usually eigenstates of the momentum operator.

$$|P^\mu, s, s_z\rangle \quad (m^2 \neq 0) \qquad \text{or} \qquad |P^\mu, \lambda\rangle \quad (m^2 = 0)\,. \tag{1.27}$$

In that sense, strictly speaking the helicity quantum number λ for massless particles is more on the same footing as the s_z quantum number for massive particles, and the analogue of s is actually $|\lambda|$. That is, commonly one would call a photon a helicity-one (or even "spin-one") particle, with the understanding that it can assume two physical polarizations with helicity ± 1.

As an aside, if the spin axis is given by a space-like vector n^μ, then a good frame-independent definition of the quantum number s_z is

$$s_z \equiv \frac{n_\mu W^\mu}{n \cdot p} \tag{1.28}$$

for states which are an eigenstate of the momentum operator.

Exercise 1: Lorentz algebra commutation relations

a. The three-dimensional complex Lie algebra sl$(2, \mathbb{C})$ is defined by the commutation relations

$$[L_i, L_j] = i\,\epsilon_{ijk} L_k \tag{1.29}$$

Show that, when regarded as a six-dimensional real Lie algebra with independent generators L_i and $K_i \equiv iL_i$, the commutation relations are those of the Lorentz algebra with the identification of Eq. (1.13).

b. Prove Eq. (1.15).

Exercise 2: Antisymmetric tensor representations

Consider the space $\mathcal{T}$ of complex-valued two-index tensors $T^{\mu\nu}$, which transform as $T^{\mu\nu} \rightarrow \Lambda^\mu_\kappa \Lambda^\nu_\lambda T^{\kappa\lambda}$ under Lorentz transformations.

a. Show that $\mathcal{T} = \mathcal{A} \oplus \mathcal{S}$ as a vector space, where $\mathcal{A}$ is the space of antisymmetric tensors (also called *two-forms*) and $\mathcal{S}$ is the space of symmetric tensors. Use this result to show that the two-index tensor representation of the Lorentz algebra is reducible.

b. Show that $\mathcal{S} = \widetilde{\mathcal{S}} \oplus \mathcal{R}$, where $\widetilde{\mathcal{S}}$ consists of traceless tensors and $\mathcal{R} = \{\lambda\, \eta^{\mu\nu} \mid \lambda \in \mathbb{C}\}$. Use this result to show that the symmetric tensor representation is reducible.

c. Using the Hodge star operator $*$, which maps an element of $\mathcal{A}$ to its *Hodge dual*,

$$(*A)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}A^{\kappa\lambda}\,, \tag{1.30}$$

show that the antisymmetric 2-tensor representation of the Lorentz group is reducible, and explain the terms "imaginary self-dual" and "imaginary anti-self-dual" in Table 1.1. (Hint: show that $*$ commutes with proper Lorentz transformations, show that it is diagonalizable, and find its eigenvalues).

d. Show that the generators $\boldsymbol{J}_{\pm}$ of Eq. (1.14) are self-dual and anti-self-dual respectively.

e. Under what conditions on the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ is the field strength tensor of electrodynamics, $F^{\mu\nu}$, self-dual?

Exercise 3: Special frames

a. Show that for a massive particle there exists a rest frame where $P^{\mu} = (m, 0, 0, 0)$.

b. Show that for a massless particle there exists a lightcone frame where $P^{\mu} = (E, 0, 0, E)$.

c. Show that for two massive particles there exists a center-of-mass frame where $P_1^{\mu} = \left(\sqrt{p^2 + m_1^2}, 0, 0, p\right)$ and $P_2^{\mu} = \left(\sqrt{p^2 + m_2^2}, 0, 0, -p\right)$.

d. Show that for two massless particles there exists a lightcone frame where $P_1^{\mu} = (E_1, 0, 0, E_1)$ and $P_1^{\mu} = (E_2, 0, 0, E_2)$.

Exercise 4: Casimir operators

a. Prove that $P^{\mu}P_{\mu}$ commutes with all Poincaré generators.

b. Work out $[W_{\mu}, P_{\nu}]$, $[W_{\mu}, M_{\kappa\lambda}]$, and $[W_{\mu}, W_{\nu}]$. Show that $W^{\mu}W_{\mu}$ commutes with all Poincaré generators.

c. Show that in a rest-frame of a massive particle W^{μ} generates the group SO(3). What physical quantity is $W^{\mu}W_{\mu}$?

d. Show that in the lightcone frame of a massless particle W^{μ} generates the group ISO(2): the group of rotations and translations in 2 dimensions.

1.2.2 Two-component spinor formalism

This Section is dedicated to the perhaps somewhat unfamiliar two-component Weyl spinor formalism. While four-component Dirac spinors are often more convenient for doing calculations in perturbative quantum field theory, Weyl spinors are much better suited for working with supersymmetry.

We have seen in the Exercises that $\mathrm{sl}(2,\mathbb{C})$ is isomorphic to the Lorentz algebra $\mathrm{so}(1,3)$ as a real Lie algebra. In fact, there is even an isomorphism of Lie groups $\mathrm{SL}(2,\mathbb{C}) \simeq \mathrm{Spin}(1,3)$: The group of 2×2 unimodular complex matrices is isomorphic to the universal cover of the proper Lorentz group $\mathrm{SO}(1,3)$.

Let $\Lambda \in \mathrm{SL}(2,\mathbb{C})$, then Λ acts on $\psi \in \mathbb{C}^2$ as

$$\psi'_\alpha = \Lambda_\alpha^{\ \beta}\psi_\beta \,. \tag{1.31}$$

This is the *left-handed Weyl spinor representation* $(\frac{1}{2},0)$. A distinct representation is provided by Λ^*:

$$\overline{\chi}'_{\dot\alpha} = \Lambda^{*}{}_{\dot\alpha}^{\ \dot\beta}\overline{\chi}_{\dot\beta} \,. \tag{1.32}$$

This is the *right-handed Weyl spinor representation* $(0,\frac{1}{2})$. To distinguish them from left-handed Weyl spinors, right-handed Weyl spinors are denoted by dotted indices.

The components of the ϵ tensors $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ are, explicitly, $\epsilon^{12}=\epsilon_{21}=1$, $\epsilon^{21}=\epsilon_{12}=-1$, and $\epsilon_{11}=\epsilon_{22}=\epsilon^{11}=\epsilon^{22}=0$. They are invariant under Lorentz transformations, for instance

$$\Lambda_\alpha^{\ \beta}\Lambda_\gamma^{\ \delta}\epsilon_{\beta\delta} = \epsilon_{\alpha\gamma} \,. \tag{1.33}$$

Therefore, if spinor indices are raised and lowered with the ϵ tensor,

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \qquad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \qquad \overline{\psi}^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\overline{\psi}_{\dot\beta}, \qquad \overline{\psi}_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\overline{\psi}^{\dot\beta}, \tag{1.34}$$

they still transform covariantly. Spinors with upper indices transform as

$$\psi'^\alpha = \Lambda^{-1}{}^{\alpha}_{\ \beta}\psi^\beta, \qquad \overline{\psi}'^{\dot\alpha} = (\Lambda^*)^{-1}{}^{\dot\alpha}_{\ \dot\beta}\overline{\psi}^{\dot\beta} \tag{1.35}$$

($\rightarrow$ Exercise).

We can combine a left-handed and a right-handed Weyl spinor into a four-component object $\Psi = (\psi_\alpha, \overline{\chi}^{\dot\beta})^T$ transforming as $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$; this is the familiar *Dirac representation.*

A basis for 2×2 complex matrices is given by the σ matrices

$$\sigma^0 = -\mathbb{1}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.36}$$

Any hermitian 2×2 matrix can be written as $P = P_\mu\sigma^\mu$ with the P_μ real. If P is hermitean, then so is $P' = \Lambda P \Lambda^\dagger$ with $\Lambda \in \mathrm{SL}(2,\mathbb{C})$, and we can expand

$$\sigma^\mu P'_\mu = \Lambda\sigma^\mu P_\mu \Lambda^\dagger \,. \tag{1.37}$$

From $\det\Lambda = 1$ we see that

$$P_0^2 - \boldsymbol{P}^2 = \det P = \det P' = P_0'^2 - \boldsymbol{P}'^2 \,, \tag{1.38}$$

i.e. P_μ and P'_μ are connected by a Lorentz transformation. This shows that the index structure of σ is $\sigma^\mu_{\alpha\dot\alpha}$ with $^\mu$ a "genuine" vector index. The $\sigma^\mu_{\alpha\dot\alpha}$ can be regarded as Clebsch-Gordan coefficients in the decomposition $(\frac{1}{2},0)\otimes(0,\frac{1}{2})=(\frac{1}{2},\frac{1}{2})$.[3] They play the same role as the γ matrices for the Dirac representation; in particular,

$$\psi^\alpha \sigma^\mu_{\alpha\dot\alpha}\partial_\mu \overline{\psi}^{\dot\alpha} \tag{1.39}$$

is a Lorentz scalar.

We define the $\overline{\sigma}$ matrices to be the σ matrices with raised spinor indices:

$$\overline{\sigma}^{\mu\dot\alpha\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\alpha\dot\alpha}\,. \tag{1.40}$$

Explicitly,

$$\overline{\sigma}^0 = \sigma^0\,, \qquad \boldsymbol{\overline{\sigma}} = -\boldsymbol{\sigma}\,. \tag{1.41}$$

With the help of $\overline{\sigma}$ we can explicitly give the Lorentz generators in the right-handed and left-handed Weyl spinor representation,

$$\begin{aligned}\sigma^{\mu\nu}{}_\alpha{}^\beta &= -\frac{i}{4}(\sigma^{[\mu}\overline{\sigma}^{\nu]})_\alpha{}^\beta = -\frac{i}{4}\left(\sigma^\mu_{\alpha\dot\alpha}\overline{\sigma}^{\nu\dot\alpha\beta} - \sigma^\nu_{\alpha\dot\alpha}\overline{\sigma}^{\mu\dot\alpha\beta}\right),\\ \overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta} &= -\frac{i}{4}(\overline{\sigma}^{[\mu}\sigma^{\nu]})^{\dot\alpha}{}_{\dot\beta} = -\frac{i}{4}\left(\overline{\sigma}^{\mu\dot\alpha\alpha}\sigma^\nu_{\alpha\dot\beta} - \overline{\sigma}^{\nu\dot\alpha\alpha}\sigma^\mu_{\alpha\dot\beta}\right),\end{aligned} \tag{1.42}$$

as well as the *Weyl representation* of the Clifford algebra through the γ matrices,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}\,. \tag{1.43}$$

Lots and lots of identities relating the σ and $\overline{\sigma}$ matrices can be proved, if one is so inclined. A few are given below as exercises.

Let us finally list our conventions for index-free notation. We use the "northwest-southeast" convention for undotted spinors,

$$\psi\chi \equiv \psi^\alpha\chi_\alpha\,. \tag{1.44}$$

With these conventions we have, noting that spinors are Grassmann-valued fields whose components anticommute,

$$\psi\chi = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi\,. \tag{1.45}$$

Also

$$\overline{\psi}\overline{\chi} = \overline{\psi}_{\dot\alpha}\overline{\chi}^{\dot\alpha} = -\overline{\psi}^{\dot\alpha}\overline{\chi}_{\dot\alpha} = \overline{\chi}_{\dot\alpha}\overline{\psi}^{\dot\alpha} = \overline{\chi}\overline{\psi} \tag{1.46}$$

and

$$(\psi\chi)^\dagger = \overline{\psi}\overline{\chi} = \overline{\chi}\overline{\psi}\,. \tag{1.47}$$

Exercise 5: Sigma matrices

Show that

$$\sigma^{0i} = -\frac{i}{2}\sigma^i, \quad \overline{\sigma}^{0i} = \frac{i}{2}\sigma^i, \quad \sigma^{ij} = \overline{\sigma}^{ij} = -\frac{1}{2}\epsilon^{ijk}\sigma^k\,.$$

[3] Indeed it is possible to find any irreducible Lorentz representation in a tensor product built out of $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ factors.

Exercise 6: Fun with spinor indices

a. Verify Eq. (1.41).

b. Verify Eq. (1.35). To do so, first show that $\Lambda^{-1} = -\epsilon\,\Lambda^T\epsilon$ for $\Lambda \in \mathrm{SL}(2,\mathbb{C})$.

c. For a constant left-handed spinor θ and its right-handed conjugate $\bar{\theta}$, show

$$\begin{aligned}
\theta^\alpha\theta^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta\,,\\
\theta_\alpha\theta_\beta &= \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta\,,\\
\bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} &= \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}\,,\\
\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} &= -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}\,,\\
\theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta} &= -\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\eta^{\mu\nu}\,.
\end{aligned}$$

1.2.3 The supersymmetry algebra

We are now finally equipped with the tools to introduce supersymmetry. Our starting point is the question whether it is possible to non-trivially extend the Poincaré algebra. That is, can we supplement the Poincaré generators with additional generators which do not commute with the P_μ or the $M_{\kappa\lambda}$?

That this is not entirely straightforward is evident from the *Coleman-Mandula theorem* which roughly states:

The symmetries of the S-matrix of any local relativistic quantum field theory are generated by energy-momentum P_μ, Lorentz transformation generators $M_{\mu\nu}$, and a finite number of Lorentz scalar operators B_A which commute with P_μ and generate a compact Lie group.

There are a number of additional assumptions which go into this theorem, such as the existence of massive particles (if no massive particles are present, then the Poincaré group may be extended to the conformal group; more on this later), that space-time is four-dimensional (the four-dimensional Poincaré group may be embedded into higher-dimensional space-time symmetry groups, which also amounts to non-trivially extending the algebra) and so on. The Coleman-Mandula theorem should be treated in most cases as a typical high-energy physicist's theorem: there are ways around it, but usually these lead to unappealingly involved constructions. The one known exception in this particular case is the subject of these lectures.

Among the assumptions, the crucial one for us is that the symmetry generators should form a Lie algebra, in particular that they should close under the commutator. The Poincaré generators P_μ, $M_{\mu\nu}$ as well as the internal symmetry generators B_A are subject to their respective defining *commutation relations*, which is of course due to their bosonic nature. It seems natural to try and extend the Poincaré algebra by charges which transform as spinors, and satisfy certain *anticommutation relations* instead. This circumvents the Coleman-Mandula theorem, since instead of a Lie algebra, one then obtains a so-called ($\mathbb{Z}_2$-)*graded Lie algebra*: the *supersymmetry algebra*.

To be concrete, let us supplement the Poincaré algebra with $\mathcal{N}$ fermionic symmetry generators Q^L in the $(\frac{1}{2},0)$ Lorentz representation and their hermitian conjugates $\overline{Q}_M$ in the $(0,\frac{1}{2})$ representation.[4] We impose the *graded Jacobi identity*:

$$\{R,\{S,T]] \pm \{S,\{T,R]] \pm \{T,\{R,S]] = 0\,. \tag{1.48}$$

Here R,S,T are any generators, and $\{\cdot,\cdot]$ stands for either the anticommutator $\{\cdot,\cdot\}$ if both arguments are fermionic, or the commutator $[\cdot,\cdot]$ otherwise. The signs are determined by the order of the fermionic elements: interchanging two fermions with respect to the first term picks up a minus sign. Explicitly, for bosonic B and fermionic F,

$$\begin{aligned}
[B,[B',B'']]+[B'',[B,B']]+[B',[B'',B]] &= 0\,,\\
[F,[B,B']]+[B',[F,B]]+[B,[B',F]] &= 0\,,\\
\{F,[F',B]\}+[B,\{F,F'\}]-\{F',[B,F]\} &= 0\,,\\
[F,\{F',F''\}]+[F'',\{F,F'\}]+[F',\{F'',F\}] &= 0\,.
\end{aligned} \tag{1.49}$$

The fermionic Q and $\overline{Q}$ generators are called *supercharges*. Note that in general the Jacobi identities should hold for a *finite* set of generators: otherwise acting with combinations of symmetry generators yields new symmetry generators ad infinitum. Typically, this puts so many constraints on a scattering process that no physics can be described by them.

The Coleman-Mandula theorem together with the Jacobi identity restrict the possible (anti-)commutation relations between the supercharges and the other Lorentz generators. In addition one demands that the supercharges should act on a Hilbert space with a positive definite and non-degenerate metric,

$$\langle\Psi|\{Q,Q^\dagger\}|\Psi\rangle \geq 0, \qquad \langle\Psi|\{Q,Q^\dagger\}|\Psi\rangle = 0 \text{ for all } \Psi \text{ only if } Q=0\,. \tag{1.50}$$

The resulting restrictions nearly completely fix the supersymmetry algebra, a result known as the *Haag-Lopuszański-Sohnius theorem.* Let us work out the (anti-)commutators explicitly.

Supercharge – conjugate supercharge: Since $(\frac{1}{2},0)\otimes(0,\frac{1}{2})=(\frac{1}{2},\frac{1}{2})$, the anticommutator of Q and $\overline{Q}$ must close into a Lorentz vector, and by the Coleman-Mandula theorem P_μ is the only possibility:

$$\{Q^L_\alpha,\overline{Q}_{\dot\alpha M}\} = \sigma^\mu_{\alpha\dot\alpha}P_\mu\, C^L{}_M\,. \tag{1.51}$$

The matrix $C^L{}_M$ is hermitian (→ Exercises) and can therefore be chosen diagonal. Furthermore, since the anticommutator should be positive definite by Eq. (1.50), the generators can be rescaled such that

$$\{Q^L_\alpha,\overline{Q}_{\dot\alpha M}\} = 2\,\sigma^\mu_{\alpha\dot\alpha}P_\mu\,\delta^L{}_M\,. \tag{1.52}$$

Supercharge – Lorentz generator: Since the Q transform as $(\frac{1}{2},0)$ and the $\overline{Q}$ transform as $(0,\frac{1}{2})$ under Lorentz transformations,

$$[M^{\mu\nu},Q^L_\alpha] = \sigma^{\mu\nu}{}_\alpha{}^\beta Q^L_\beta, \qquad [\overline{Q}^M_{\dot\alpha},M^{\mu\nu}] = \overline{Q}^M_{\dot\beta}\,\overline{\sigma}^{\mu\nu\dot\beta}{}_{\dot\alpha}\,. \tag{1.53}$$

[4] Using the Coleman-Mandula theorem it can be shown that any fermionic generators, if present, cannot transform under representations of higher spin.

Supercharge − momentum: Since $(\frac{1}{2},0)\otimes(\frac{1}{2},\frac{1}{2})=(1,\frac{1}{2})\oplus(0,\frac{1}{2})$ and there are no $(1,\frac{1}{2})$ generators, the commutator $[Q^L_\alpha, P^\mu]$ must take the form

$$[Q^{\alpha L}, P^\mu] = Z^L{}_M \overline{\sigma}^{\mu\alpha\dot{\alpha}} \overline{Q}^M_{\dot{\alpha}} \tag{1.54}$$

for some matrix $Z^L{}_M$. Therefore

$$[\overline{Q}_{\dot{\alpha}L}, P^\mu] = Z^{*M}{}_L\, Q^\alpha_M\, \sigma^\mu_{\alpha\dot{\alpha}}\,. \tag{1.55}$$

Since $[P_\mu, P_\nu]=0$, the Jacobi identity yields

$$0 = [[Q^{\alpha L}, P^\nu], P^\mu] - [[Q^{\alpha L}, P^\mu], P^\nu] = 4i\,\sigma^{\mu\nu\alpha}_\beta Z^L{}_M Z^{*M}{}_K Q^{\beta K} \tag{1.56}$$

which implies $(ZZ^*)^L_K = 0$ and thus

$$[Q^L_\alpha, P^\mu] = [\overline{Q}^M_{\dot{\alpha}}, P^\mu] = 0\,. \tag{1.57}$$

Supercharge − supercharge: Since $(\frac{1}{2},0)\otimes(\frac{1}{2},0)=(0,0)\oplus(1,0)$, we have

$$\{Q^L_\alpha, Q^M_\beta\} = \epsilon_{\alpha\beta} X^{LM} + \sigma^{\mu\nu}_{\alpha\beta}\, M_{\mu\nu} Y^{LM} \tag{1.58}$$

where X is an antisymmetric matrix and Y is a symmetric matrix. Using $[P_\mu, Q^L_\alpha]=0$ and the Jacobi identity, we find

$$0 = [P_\mu, \{Q^L_\alpha, Q^M_\beta\}] = [P_\mu, \sigma^{\kappa\lambda}_{\alpha\beta} M_{\kappa\lambda} Y^{LM}] \tag{1.59}$$

and therefore $Y^{LM}=0$. We conclude that

$$\{Q^L_\alpha, Q^M_\beta\} = \epsilon_{\alpha\beta} X^{LM} = \epsilon_{\alpha\beta} a^{A\,LM} R_A\,, \tag{1.60}$$

with R_A generating an internal symmetry which commutes with the Poincaré generators,

$$[R_A, R_B] = i\, f_{AB}{}^C R_C\,, \qquad [R_A, P_\mu] = 0, \qquad [R_A, M_{\mu\nu}] = 0\,. \tag{1.61}$$

The $a^{A\,LM}$ are antisymmetric in L and M.

Supercharge − R_A: We have introduced new generators of internal symmetries, whose commutators with the supercharges also need to be worked out. Since $(\frac{1}{2},0)\otimes(0,0)=(\frac{1}{2},0)$, we have

$$[Q^M_\alpha, R_A] = (S_A)^M{}_L Q^L_\alpha \tag{1.62}$$

and accordingly

$$[R_A, \overline{Q}_{\dot{\alpha}L}] = (S^*_A)_L{}^M \overline{Q}_{\dot{\alpha}M}\,. \tag{1.63}$$

From the Jacobi identity

$$[R_A, \{Q^L_\alpha, \overline{Q}_{\dot{\beta}M}\}] + \{Q^L_\alpha, [\overline{Q}_{\dot{\beta}M}, R_A]\} - \{\overline{Q}_{\dot{\beta}M}, [R_A, Q^L_\alpha]\} = 0\,. \tag{1.64}$$

Inserting Eqns. (1.52), (1.62) and (1.63) we find

$$2\, P_\mu \sigma^\mu_{\alpha\dot{\beta}} \left((S^*_A)_M{}^L - (S_A)^L{}_M\right) = 0 \tag{1.65}$$

which shows that S_A is hermitian. The generators R_A are thus seen to act by a unitary representation on the $\mathcal{N}$-dimensional space of supersymmetry generators Q_L. Those R_A which are nontrivially represented are said to generate an *R-symmetry*.

Central charges: It can further be shown that the linear combinations

$$c^{MN} = a^{A\,MN} R_A \tag{1.66}$$

commute with all internal symmetry generators, as well as with the supercharges and with each other. They thus generate an abelian subalgebra of the internal symmetry algebra and are called *central charges*. Central charges will be important in the context of extended supersymmetry, and we will revisit them in more detail later.

Let us summarize: The supersymmetry algebra consists of the Poincaré algebra as defined by Eqns. (1.5) and (1.16), supplemented by $\mathcal{N}$ left-handed Weyl spinor generators Q^L_α, $\mathcal{N}$ right-handed Weyl spinor generators $\overline{Q}_{\dot{\alpha}M}$, and R-symmetry generators R_A. They satisfy the commutation relations

$$\boxed{\begin{aligned}
\{Q^L_\alpha, \overline{Q}_{\dot{\alpha}M}\} &= 2\,\sigma^\mu_{\alpha\dot{\alpha}} P_\mu\, \delta^L{}_M\,, \\
\{Q^L_\alpha, Q^M_\beta\} &= \epsilon_{\alpha\beta} a^{A\,LM} R_A\,, \\
\{\overline{Q}_{\dot{\alpha}L}, \overline{Q}_{\dot{\beta}M}\} &= \epsilon_{\dot{\alpha}\dot{\beta}} a^{*A}{}_{LM} R_A\,, \\
[M^{\mu\nu}, Q^L_\alpha] &= \sigma^{\mu\nu}{}_\alpha{}^\beta Q_\beta^{\cdot}, \\
[\overline{Q}^M_{\dot{\alpha}}, M^{\mu\nu}] &= \overline{Q}^M_{\dot{\beta}}\, \overline{\sigma}^{\mu\nu\dot{\beta}}{}_{\dot{\alpha}}\,, \\
[P_\mu, Q^L_\alpha] &= [P_\mu, \overline{Q}_{\dot{\alpha}L}] = 0\,, \\
[R_A, R_B] &= i\, f_{AB}{}^C R_C\,, \\
[Q^M_\alpha, R_A] &= (S_A)^M{}_L Q^L_\alpha\,, \\
[R_A, \overline{Q}_{\dot{\alpha}L}] &= (S^*_A)_L{}^M \overline{Q}_{\dot{\alpha}M}\,, \\
[c^{LM}, Q^N_\alpha] &= [c^{LM}, \overline{Q}_{\dot{\alpha}N}] = [c^{LM}, c^{NK}] = [c^{LM}, R_A] = 0 \\
&\qquad (\text{where } c^{MN} = a^{A\,MN} R_A)\,.
\end{aligned}} \tag{1.67}$$

Exercise 7: SUSY algebra commutation relations

a. Show that the matrix $C^L{}_M$ in Eq. (1.51) is hermitian.

b. Use the Jacobi identity for two R_A generators and a supercharge to show that

$$[S_A, S_B] = i\, f_{AB}{}^C S_C$$

where the matrices $(S_A)^M{}_L$ are defined in Eq. (1.62).

1.2.4 Representations of the supersymmetry algebra

The supercharge Q_α applied to a bosonic state gives a fermionic state, and vice versa. Every representation of the supersymmetry algebra contains an equal number of bosonic and fermionic states. This is proved as follows: If N_f counts the fermion number, such that for any bosonic state $|B\rangle$ and any fermionic state $|F\rangle$

$$(-)^{N_F}|B\rangle = +|B\rangle, \qquad (-)^{N_F}|F\rangle = -|F\rangle\,, \tag{1.68}$$

then

$$(-)^{N_F} Q_\alpha = -Q_\alpha (-)^{N_F} \tag{1.69}$$

and therefore, for any (finite-dimensional) representation,

$$\mathrm{tr}\left((-)^{N_F}\{Q^L_\alpha, \overline{Q}_{\dot\beta M}\}\right) = \mathrm{tr}\left(-Q^L_\alpha(-)^{N_F}\overline{Q}_{\dot\beta M} + Q^L_\alpha(-)^{N_F}\overline{Q}_{\dot\beta M}\right) = 0\,. \qquad (1.70)$$

Thus, by the supersymmetry algebra,

$$2\,\sigma^\mu_{\alpha\dot\beta}\delta^L_M\,\mathrm{tr}\left((-)^{N_F}P_\mu\right) = 0\,, \qquad (1.71)$$

and therefore, for fixed non-zero momentum P_μ,

$$\mathrm{tr}(-)^{N_F} = 0 \qquad (1.72)$$

which shows that every supersymmetry representation contains an equal number of fermionic and bosonic states.

We proceed to classify the representations of the SUSY algebra for the case without central charges.[5] Since $P^\mu P_\mu$ commutes with all Poincaré generators and with the supercharges, it is a Casimir operator of the supersymmetry algebra. However $W^\mu W_\mu$ can obviously not be a Casimir operator since the states in a supersymmetry multiplet have different spins. As for the Poincaré algebra, we distinguish between massless and massive representations.

1. $m = 0$. We boost to the light-cone frame, where $P^\mu = (E, 0, 0, E)$ and where

$$\{Q^L_\alpha, \overline{Q}_{\dot\beta M}\} = 2\begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}\delta^L{}_M, \qquad \{Q,Q\} = \{\overline{Q},\overline{Q}\} = 0\,. \qquad (1.73)$$

The $2\mathcal{N}$ generators Q^L_2 and $\overline{Q}_{2M}$ anticommute with everything, and must therefore vanish by Eq. (1.50). The $2\mathcal{N}$ rescaled generators

$$a^L = \frac{1}{2\sqrt{E}}Q^L_1, \qquad a^\dagger_L = \frac{1}{2\sqrt{E}}\overline{Q}_{1L} \qquad (1.74)$$

satisfy an algebra of fermionic creation and annihilation operators:

$$\{a^L, a^\dagger_M\} = \delta^L{}_M\,, \qquad \{a^L, a^M\} = \{a^\dagger_L, a^\dagger_M\} = 0\,. \qquad (1.75)$$

The representations of this algebra are created from some Clifford ground state, which is characterized by its energy E and helicity λ_0, and which is annihilated by a:

$$a|E,\lambda_0\rangle = 0\,. \qquad (1.76)$$

By applying the $a^\dagger_M$ to the ground state one finds other states

$$\frac{1}{\sqrt{n!}}\,a^\dagger_{M_1}\cdots a^\dagger_{M_n}|E,\lambda_0\rangle\,.$$

They are antisymmetric in $M_1 \ldots M_n$ because the $a^\dagger_M$ anticommute, so their degeneracy is $\binom{\mathcal{N}}{n}$. Their helicity is $\lambda_0 + \frac{n}{2}$ because $a^\dagger_M$ raises the helicity by $\frac{1}{2}$. This is seen as follows: We have, with the helicity operator $L = M_{12}$ of Eq. (1.23) represented by $\overline{\sigma}^{12} = -\frac{1}{2}\sigma^3$,

$$L\,a^\dagger_M|E,\lambda\rangle = [L, a^\dagger_M]|E,\lambda\rangle + a^\dagger_M\lambda|E,\lambda\rangle = \left(\tfrac{1}{2}+\lambda\right)a^\dagger_M|E,\lambda\rangle \qquad (1.77)$$

[5]See e.g. the book by Wess and Bagger for the general case.

where we have used that $(a^\dagger_M, 0)\sigma^3 = (a^\dagger, 0)$ and Eq. (1.53). Similarly, the a_M lower the helicity by $\frac{1}{2}$. The state of highest helicity

$$a^\dagger_1 \cdots a^\dagger_{\mathcal{N}} |E, \lambda_0\rangle = |E, \lambda_0 + \tfrac{\mathcal{N}}{2}, 1 \ldots \mathcal{N}\rangle \tag{1.78}$$

is annihilated by all $a^\dagger_M$. The dimension of the representation is

$$d = \sum_{i=1}^{\mathcal{N}} \binom{\mathcal{N}}{i} = 2^{\mathcal{N}} \,. \tag{1.79}$$

For example, if there is only $\mathcal{N} = 1$ supercharge, the massless supersymmetry representation built from a Clifford ground state of helicity $\lambda_0 = 0$ contains the states

$$|E, 0\rangle, \qquad a^\dagger |E, 0\rangle = |E, \tfrac{1}{2}\rangle \,. \tag{1.80}$$

Indeed for $\mathcal{N} = 1$ any massless multiplet contains precisely two states with their helicities differing by $\frac{1}{2}$. For $\mathcal{N} = 2$, any massless multiplet contains one state with helicity λ_0, two with $\lambda_0 + \frac{1}{2}$, and one with $\lambda_0 + 1$. For $\mathcal{N} = 4$, one has one state for each helicity λ_0 and $\lambda_0 + 2$, four states for each helicity $\lambda_0 + \frac{1}{2}$ and $\lambda_0 + \frac{3}{2}$, and $\binom{4}{2} = 6$ states of helicity $\lambda_0 + 1$.

2. $m > 0$. We boost to the rest frame, where $P^\mu = (m, 0, 0, 0)$ and where

$$\{Q^L_\alpha, \overline{Q}_{\dot\beta M}\} = 2\, m\, \delta_{\alpha\dot\beta} \delta^L{}_M, \qquad \{Q^L_\alpha, \overline{Q}^M_\beta\} = \{\overline{Q}_{\dot\alpha L}, \overline{Q}_{\dot\beta M}\} = 0 \,. \tag{1.81}$$

The $4\mathcal{N}$ rescaled generators

$$a^L_\alpha = \frac{1}{\sqrt{2m}} Q^L_\alpha, \qquad (a^L_\alpha)^\dagger = \frac{1}{\sqrt{2m}} \overline{Q}_{\dot\alpha L} \tag{1.82}$$

satisfy the algebra of $2\mathcal{N}$ pairs of fermionic creation and annihilation operators:

$$\{a^L_\alpha, (a^M_\beta)^\dagger\} = \delta^\beta{}_\alpha \delta^L{}_M \,, \qquad \{a, a\} = \{a^\dagger, a^\dagger\} = 0 \,. \tag{1.83}$$

The representations of this algebra are constructed by acting with the creation operators $a^\dagger$ on some spin multiplet of Clifford ground states of mass m and spin s_0. The ground state multiplet is $(2s_0 + 1)$-fold degenerate, with the degeneracy labelled by a quantum number $s_3 = -s_0, -s_0 + 1, \ldots, s_0 - 1, s_0$. Again the ground states are annihilated by all a:

$$a^L_\alpha |m, s_0, s_3\rangle = 0 \,. \tag{1.84}$$

The state

$$\frac{1}{\sqrt{n!}} (a^{M_1}_{\alpha_1})^\dagger \cdots (a^{M_n}_{\alpha_n})^\dagger |m, s_0, s_3\rangle$$

is antisymmetric under the exchange of index pairs $(\alpha_i, M_i) \leftrightarrow (\alpha_j, M_j)$, and therefore $\binom{2\mathcal{N}}{n}$-fold degenerate. The maximal spin is carried by states like

$$(a^1_1)^\dagger (a^2_1)^\dagger \cdots (a^{\mathcal{N}}_1)^\dagger |m, s_0, s_3\rangle \tag{1.85}$$

which are maximally symmetric in the spinor indices. It is given by $s_{\max} = s_0 + \mathcal{N}/2$. The minimal spin is $s_0 - \mathcal{N}/2$ (or 0 if $s_0 < \mathcal{N}/2$).

$\mathcal{N}=4$:

spin	$s_0=0$
0	42
$\frac{1}{2}$	48
1	27
$\frac{3}{2}$	8
2	1

$\mathcal{N}=2$:

spin	$s_0=0$	$s_0=\frac{1}{2}$	$s_0=1$
0	5	4	1
$\frac{1}{2}$	4	6	4
1	1	4	6
$\frac{3}{2}$		1	4
2			1

$\mathcal{N}=1$:

spin	$s_0=0$	$s_0=\frac{1}{2}$	$s_0=1$	$s_0=\frac{3}{2}$
0	2	1		
$\frac{1}{2}$	1	2	1	
1		1	2	1
$\frac{3}{2}$			1	2
2				1

Table 1.2: Multiplicities of states in the massive multiplets for $\mathcal{N}=4$ (top left), $\mathcal{N}=2$ (top right) and $\mathcal{N}=1$ (bottom).

A particularly important example is the *fundamental massive multiplet* which is obtained by acting on a spin-0 ground state. For $\mathcal{N}=1$ this representation contains only three states, one of spin-$\frac{1}{2}$ and two of spin-0:

$$|m,0,0\rangle\,, \qquad a^\dagger_\alpha|m,0,0\rangle\,, \qquad \tfrac{1}{\sqrt{2}}a^\dagger_1 a^\dagger_2|m,0,0\rangle\,.$$

In the next section, we will see a field-theoretic realization of this multiplet as a complex scalar and a Weyl spinor, constituting the simplest supersymmetric model. Massive representations for $\mathcal{N}=1$, 2 and 4 are listed in Table 1.2.

A few remarks on minimal and maximal representations are in order. First, as previously stated, CPT invariance demands that states with helicity λ be accompanied by states with helicity $-\lambda$. The smallest field content of a supersymmetric theory is therefore two bosonic and two fermionic degrees of freedom in $\mathcal{N}=1$. Either these are contained in a fundamental massive $\mathcal{N}=1$ multiplet, or they form two massless representations with Clifford ground states of helicities $\lambda_0=0$ and $\lambda_0=-\frac{1}{2}$.

Second, on rather general grounds it can be shown that four-dimensional field theories cannot be consistently coupled to gravity if they contain massless fields of helicity larger than 2. This implies that the maximal number of supercharges in four dimensions is $\mathcal{N}=8$. Furthermore, an $\mathcal{N}=8$ supersymmetric theory can contain just a single multiplet (the gravitational multiplet) with states of helicity ±2, $\pm\frac{3}{2}$ ($\times8$), ±1 ($\times28$), $\pm\frac{1}{2}$ ($\times56$), and 0 ($\times70$). Because it contains a massless spin-2 particle (a graviton), it is necessarily a theory of supergravity.

Of particular interest is also the maximally supersymmetric renormalizable theory. Renormalizable theories can only contain states of helicity up to one (scalars, spinors, and vector bosons). The maximally supersymmetric renormalizable theory therefore has $\mathcal{N}=4$ and a single type of massless multiplet, containing states of helicity ±1 (gauge bosons), $\pm\frac{1}{2}$ ($\times4$), and 0 ($\times6$). This is $\mathcal{N}=4$ super-Yang-Mills theory.

Exercise 8: Massive SUSY representations

a. Convince yourself that the representations in Table 1.2 contain an equal number of bosonic and fermionic degrees of freedom.

b. Prove that the highest spin in the fundamental massive multiplet occurs exactly once. Construct the state with the highest spin in the z-direction.

1.3 Supersymmetric field theory

The basic ingredients of a quantum field theory are fields. Just as above we studied representations of supersymmetry on states, here we would like to study fields which transform under the action of the supersymmetry algebra. Fields are operators $\Phi(x)$ which generate states from the vacuum such as

$$|x\rangle = \Phi(x)|0\rangle \,. \tag{1.86}$$

For illustration, it is useful to recall how ordinary space-time symmetries act on such a state. Consider for instance a translation which is generated by the momentum operator:

$$|x+y\rangle = e^{iy^\mu P_\mu}|x\rangle \,, \tag{1.87}$$

which implies

$$\Phi(x+y) = e^{iy^\mu P_\mu}\Phi(x)e^{-iy^\mu P_\mu} = \Phi(x) + iy^\mu[P_\mu, \Phi(x)] + \mathcal{O}(|y|^2) \,. \tag{1.88}$$

Comparing the RHS with the expansion $\Phi(x+y) = \Phi(x) + y^\mu\partial_\mu\Phi(x) + \mathcal{O}(|y|^2)$ we recover the familiar result that the momentum operator is realized on the field Φ as

$$[P_\mu, \Phi] = -i\partial_\mu\Phi \,. \tag{1.89}$$

We seek a similar representation of the supercharges on fields. In the next chapter we will indeed see how to realize them as differential operators; for now we will use the known action of the momentum operator to find the degrees of freedom in a simple supermultiplet, and their supersymmetry transformations.

A difference between on- and off-shell

The representations discussed above were on their mass-shell ('on-shell'): the momentum squared was fixed, $p^2 = m^2$. Fields however generically do not obey their equations of motion: they are off their mass-shell ('off-shell'). There are several differences between these cases which can be illustrated by simple counting. If the fields are massless the field content of the simplest *on-shell* multiplet obtained above is one complex scalar and one on-shell Weyl fermion which also has one complex degree of freedom. Off-shell however these degrees of freedom are given by two fields,

$$\psi_\alpha(x) \,, \qquad \phi(x) \,.$$

For CPT symmetry we should also include the conjugate multiplet,

$$\overline{\psi}_{\dot\alpha}(x) \,, \qquad \phi^*(x) \,.$$

In the massive case the two Weyl fermions combine to give an on-shell Dirac fermion (two complex degrees of freedom). The attentive reader will have noticed that this off-shell field content does not satisfy the "bosons = fermions" rule! Off-shell the Weyl spinor field has two degrees of freedom, while the scalar field only has one. To make the counting work we would need two complex scalar fields F, F^*, one for each multiplet. However, these cannot allowed to be propagating, dynamical fields: they should only obey algebraic equations of motion (no derivatives). Fields of this type are called auxiliary. This mismatch between on- and off-shell degrees of freedom is a generic feature of supersymmetric field theories.

An alternative approach is to forsake the use of auxiliary fields. One can still construct a supersymmetric field theory, but the supersymmetry algebra will only hold up to terms which are proportional to field equations. In practical calculations this means that supersymmetry in this approach will not be manifest in intermediate stages of a calculation (say in individual Feynman graphs).

1.3.1 A simple supersymmetric field theory

We start with a scalar field $\phi(x)$ and demand that it should be annihilated by $\overline{Q}$,

$$\boxed{[\overline{Q}_{\dot\alpha}, \phi] = 0\,.} \tag{1.90}$$

For now there is no deeper justification for this *chiral constraint* except that it will lead us to a sensible multiplet. By the Jacobi identity,

$$\{[Q,\phi],\overline{Q}\} + \{Q,[\overline{Q},\phi]\} = [\{Q,\overline{Q}\},\phi] = -2i\,\sigma^\mu \partial_\mu \phi \tag{1.91}$$

ϕ is a complex field (otherwise the LHS would vanish, so ϕ would have to be constant).

We now define a field $\psi_\alpha(x)$ by

$$\boxed{[Q_\alpha, \phi] = \sqrt{2}\,\psi_\alpha\,.} \tag{1.92}$$

Inserting the chirality constraint Eq. (1.90) into the Jacobi identity Eq. (1.91) gives

$$\sqrt{2}\{\psi_\alpha, \overline{Q}_{\dot\alpha}\} = -2i\,\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\phi \qquad \Rightarrow \qquad \{\psi_\alpha, \overline{Q}_{\dot\alpha}\} = -\sqrt{2}i\,\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\phi\,. \tag{1.93}$$

The Jacobi identity for two supercharges and ϕ reads

$$\begin{aligned} 0 =& -[\phi,\{Q_\alpha,Q_\beta\}] = \{Q_\alpha,[Q_\beta,\phi] - \{Q_\beta,[\phi,Q_\alpha]\}\} \\ =&\sqrt{2}\left(\{Q_\alpha,\psi_\beta\} + \{Q_\beta,\psi_\alpha\}\right)\,. \end{aligned} \tag{1.94}$$

Therefore $\{Q_\alpha, \psi_\beta\}$ is antisymmetric in α and β, and we must have

$$\boxed{\{Q_\alpha, \psi_\beta\} = -\sqrt{2}\,\epsilon_{\alpha\beta}\,F} \tag{1.95}$$

with some complex scalar field $F(x)$.

It turns out that $\phi(x)$, $\psi(x)$ and $F(x)$ are all the fields we need to construct a supermultiplet, because the other Jacobi identities only lead to relations between ϕ, ψ and F. Let us work these out.

The Jacobi identity

$$[\{\psi_\alpha, Q_\beta\}, \overline{Q}_{\dot\beta}] = [\psi_\alpha, \{Q_\beta, \overline{Q}_{\dot\beta}\}] - [\{\psi_\alpha, \overline{Q}_{\dot\beta}\}, Q_\beta] \tag{1.96}$$

yields the condition

$$\sqrt{2}\,\epsilon_{\alpha\beta}[F, \overline{Q}_{\dot\beta}] = [\psi_\alpha, 2\sigma^\mu_{\beta\dot\beta}P_\mu] - 2i\sigma^\mu_{\alpha\dot\beta}[\partial_\mu\phi, Q_\beta] = 2i\left(\sigma^\mu_{\beta\dot\beta}\partial_\mu\psi_\alpha - \sigma^\mu_{\alpha\dot\beta}\partial_\mu\psi_\beta\right) . \tag{1.97}$$

Contraction with $\epsilon^{\beta\alpha}$ yields

$$\begin{aligned} 2\sqrt{2}\,[F, \overline{Q}_{\dot\beta}] = 2i\left(\sigma^\mu_{\beta\dot\beta}\partial_\mu\psi^\beta + \sigma^\mu_{\alpha\dot\beta}\partial_\mu\psi^\alpha\right) &= 4i\,\sigma^\mu_{\alpha\dot\beta}\partial_\mu\psi^\alpha \\ \Rightarrow \qquad [F, \overline{Q}_{\dot\beta}] &= \sqrt{2}i\,\sigma^\mu_{\alpha\dot\beta}\partial_\mu\psi^\alpha\,. \end{aligned} \tag{1.98}$$

This can also be written in the slightly nicer form

$$[\overline{Q}^{\dot\alpha}, F] = \sqrt{2}i\,\overline{\sigma}^{\mu\dot\alpha\alpha}\partial_\mu\psi_\alpha\,. \tag{1.99}$$

Similarly,

$$0 = [\{\psi_\alpha, Q_\beta\}, Q_\gamma] + [\{Q_\gamma, \psi_\alpha\}, Q_\beta] + [\{Q_\beta, Q_\gamma\}, \psi_\alpha] \tag{1.100}$$

leads to the condition

$$\epsilon_{\alpha\beta}[F, Q_\gamma] + \epsilon_{\alpha\gamma}[F, Q_\beta] = 0 \qquad \Rightarrow \qquad [F, Q_\alpha] = 0 \tag{1.101}$$

where the last implication again follows from contracting with $\epsilon^{\beta\alpha}$.

Finally, it can be checked that the remaining Jacobi identities are satisfied and give no new information. Summarizing, we have found a multiplet of fields which carries a representation of the supersymmetry algebra:

$$\boxed{\begin{aligned} [Q_\alpha, \phi] &= \sqrt{2}\,\psi_\alpha\,, & [\overline{Q}_\alpha, \phi] &= 0\,, \\ \{Q_\alpha, \psi_\beta\} &= -\sqrt{2}\,\epsilon_{\alpha\beta}F\,, & \{\psi_\alpha, \overline{Q}_{\dot\alpha}\} &= -\sqrt{2}i\,\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\phi\,, \\ [Q_\alpha, F] &= 0\,, & [\overline{Q}^{\dot\alpha}, F] &= \sqrt{2}i\,\overline{\sigma}^{\mu\dot\alpha\alpha}\partial_\mu\psi_\alpha\,. \end{aligned}} \tag{1.102}$$

With the help of Grassmann-valued (anticommuting) transformation parameters ξ^α, $\bar\xi_{\dot\alpha}$, we can define the *supersymmetry variation* of a field $\Phi(x)$ under a transformation with parameter ξ:

$$\delta_\xi\Phi(x) = [(\xi Q + \overline{\xi Q}), \Phi(x)]\,. \tag{1.103}$$

This is completely in analogy to ordinary bosonic symmetry transformations with c-number parameters. The supersymmetry variations

$$\boxed{\begin{aligned} \delta_\xi\phi &= \sqrt{2}\xi\,\psi\,, \\ \delta_\xi\psi &= \sqrt{2}i\,\sigma^\mu\bar\xi\partial_\mu\phi + \sqrt{2}\xi\,F\,, \\ \delta_\xi F &= \sqrt{2}i\,\bar\xi\overline{\sigma}^\mu\partial_\mu\psi\,, \end{aligned}} \tag{1.104}$$

are easily checked to be completely equivalent to Eqns. (1.102). The SUSY variations can be shown to close (into the Poincaré algebra) in the sense that

$$(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\Phi(x) = -2i\,(\eta\sigma^\mu\bar\xi - \xi\sigma^\mu\overline{\eta})\,\partial_\mu\Phi(x)\,. \tag{1.105}$$

Note that, since Q has mass dimension 1/2 and taking ϕ to have canonical dimension 1, the mass dimensions for ψ and for F are 3/2 (which is the canonical dimension for a spinor) and 2 respectively. Summarizing, we have found what is called the *chiral multiplet*: a scalar $\phi(x)$, a spinor $\psi(x)$, and a dimension-2 scalar $F(x)$ which transform as in Eqns. (1.102) or equivalently Eqns. (1.104) .

To find a Lagrangian, one needs to find combinations of fields which transform into total derivatives. For a Lagrangian quadratic in the fields, the unique choice is

$$\mathcal{L} = i\partial_\mu\overline{\psi}\overline{\sigma}^\mu\psi - \partial_\mu\phi^*\partial^\mu\phi + F^*F + m\left(\phi F + \phi^* F^* - \frac{1}{2}\psi\psi - \frac{1}{2}\overline{\psi\psi}\right) \tag{1.106}$$

where m is a mass parameter. This is a *free massive Wess-Zumino model*. The field equations are

$$i\overline{\sigma}^\mu\partial_\mu\psi + m\overline{\psi} = 0\,, \qquad F + m\phi^* = 0\,, \qquad \partial_\mu\partial^\mu\phi + mF^* = 0\,. \tag{1.107}$$

Note that the field equation for F is algebraic: One cannot write down a dimension-four kinetic term for the dimension-two scalar F. An equivalent Lagrangian is therefore given by substituting F and F^* via their equations of motion, $F \to -m\phi^*$ and $F^* \to m\phi$, which leads to

$$\mathcal{L} = i\partial_\mu\overline{\psi}\overline{\sigma}^\mu\psi - \partial_\mu\phi^*\partial^\mu\phi - m^2|\phi|^2 - \frac{1}{2}\psi\psi - \frac{1}{2}\overline{\psi\psi} \tag{1.108}$$

with the remaining field equations being the ordinary Weyl and Klein-Gordon equations,

$$i\overline{\sigma}^\mu\partial_\mu\psi + m\overline{\psi} = 0\,, \qquad \left(\partial_\mu\partial^\mu\phi - m^2\right)\phi = 0\,. \tag{1.109}$$

They describe a complex scalar and a Weyl spinor of mass m. However, as already stressed, if F is integrated out then the Lagrangian Eq. (1.108) is supersymmetry invariant only when the equations of motion hold. For supersymmetry to hold *off-shell* one needs the auxiliary field F, even though it contains no propagating degrees of freedom.

1.3.2 Superspace and superfields

The preceding section has shown our very first supersymmetric field theory. From the construction it should be clear that the general construction of supersymmetric field theories can become cumbersome quickly. Generically in physics symmetry leads to simple expressions only when the right coordinates are introduced. Just think about the orbit of the earth around the sun in Cartesian and in spherical coordinates. This leads to the question what the right coordinates are. Bosonic space-time symmetries are intimately connected to functions on space-time itself, $\mathbb{R}^{1,3}$: the reason one always writes an action as

$$S = \int \mathrm{d}^4x\,\mathcal{L}(x) \tag{1.110}$$

is that this construction guarantees a theory with Poincare symmetry for a given scalar Lagrangian. Hence we are looking for natural extension of space-time on which supersymmetry acts naturally. The extra coordinates on this space should be 'fermionic', in a sense to be made precise below: this idea leads to the notion of *superspace*. Before diving into this let us review fermionic numbers, also known as *Grassmann numbers*, first.

Grassmann numberology

Grassmann numbers are useful objects which satisfy anti-commutation relations instead of commutation relations. That is, if a and b are ordinary numbers, then these numbers commute

$$a \cdot b = b \cdot a\,, \qquad \text{normal numbers}\,. \tag{1.111}$$

For two Grassmann numbers ψ and ξ however

$$\psi \cdot \xi = -\xi \cdot \psi\,, \qquad \text{Grassmann numbers} \tag{1.112}$$

holds. Hence Grassmann numbers are 'fermionic'. In particular this means that

$$\psi \cdot \psi = \xi \cdot \xi = 0\,. \tag{1.113}$$

That implies that a function of a Grassmann number has a very simple finite Taylor series:

$$f(\psi) = f_0 + \psi f_1\,. \tag{1.114}$$

Note also that the product of two Grassmann numbers behaves as a bosonic number, i.e.

$$(\psi\xi)\eta = \eta(\psi\xi) \tag{1.115}$$

for a Grassmann number η. Grassmann numbers can be naturally differentiated by a 'fermionic' Leibniz rule,

$$\left\{\tfrac{\partial}{\partial\eta}, \eta\right\} = 1\,, \tag{1.116}$$

compared to the usual

$$\left[\tfrac{\partial}{\partial x}, x\right] = 1\,. \tag{1.117}$$

Integration is rather trivial as the rules are:

$$\int \mathrm{d}\eta\ \eta \equiv 1 \qquad \int \mathrm{d}\eta\ 1 \equiv 0\,. \tag{1.118}$$

More complicated integrals follow by the rule that integration is a linear operation

$$\int \mathrm{d}\eta\ (a\eta + b) \equiv a \qquad \forall\ a, b \in \mathbb{C}\,. \tag{1.119}$$

And finally

$$\int \mathrm{d}\eta\ \psi = 0\,. \tag{1.120}$$

The last rule makes integration invariant under fermionic shifts of the coordinate. Note that differentiation and integration are in practice very similar. One use for integration is to obtain from equation (1.114)

$$\int \mathrm{d}\eta\ f(\eta) = f_1 \qquad \text{and} \qquad \int \mathrm{d}\eta\ \eta f(\eta) = f_0\,. \tag{1.121}$$

This concludes our review of Grassmann or fermionic numbers.

Superspace

Just as vectors can be introduced for bosonic numbers, Grassmann numbers can also be generalized to objects transforming under the Lorentz group. Because of the spin-statistics theorem it is natural to introduce Grassmann coordinates which transform as left-handed and right-handed Weyl spinors $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$. The total superspace is the union of this with x^μ. Functions on this space are now given as

$$\Phi(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) \tag{1.122}$$

and are known as *superfields*. We now need to figure out the representation of the supersymmetry algebra on these functions. The simplest generator is P_μ which is simply

$$P_\mu \Phi(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) = -i \frac{\partial}{\partial x^\mu} \Phi(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) \tag{1.123}$$

since it generates translations in space (see Eqs. (1.2) and (1.89)). Similarly, one would like the supersymmetry generators to generate translations in the fermionic directions, i.e.

$$Q_\alpha \stackrel{?}{=} \frac{\partial}{\partial \theta^\alpha} \qquad \overline{Q}_{\dot{\alpha}} \stackrel{?}{=} \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} \tag{1.124}$$

However, this does not generate the right algebra, for instance $\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} \stackrel{!}{=} 2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu$. This also shows how to remedy the situation: extend the generators by

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} + \bar{\theta}^{\dot{\alpha}} \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \qquad \overline{Q}_{\dot{\alpha}} = \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - \theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \tag{1.125}$$

These are the generators of supersymmetry on superspace. These generators can be checked to generate the supersymmetry algebra. The supersymmetry generators just defined are not the only generators on this space which obey the SUSY algebra: Introducing a minus sign into one of the two generators in the guess in Eq. (1.124) leads to super-covariant derivative generators,

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} - \bar{\theta}^{\dot{\alpha}} \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \qquad \overline{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - \theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \tag{1.126}$$

These anti-commute with the SUSY generators,

$$\{Q, D\} = \{\overline{Q}, D\} = \{Q, \overline{D}\} = \{\overline{Q}, \overline{D}\} = 0 \tag{1.127}$$

Apart from guessing as we did to get to Eq. (1.125) one can also proceed more formally. For this recall from Eq. (1.87) that a field at a point x can formally be defined as[6]

$$\Phi(x) = e^{ixP}\Phi(0) \tag{1.128}$$

Similarly, we then also have

$$\Phi(x, \theta, \bar{\theta}) = e^{ixP + i\theta Q + i\bar{\theta}\bar{Q}}\Phi(0, 0, 0) \tag{1.129}$$

A finite supersymmetry transformation acts on this field as

$$e^{i\xi Q + i\bar{\xi}\bar{Q}} e^{ixP + \theta Q + \bar{\theta}\bar{Q}}\Phi(0) = \Phi(x + 2i\xi\sigma\bar{\xi}, \theta + \xi, \bar{\theta} + \bar{\xi}) \tag{1.130}$$

[6]Note this is a coherent state for the harmonic oscillator described by the Leibniz rule equation (1.117): it is an eigenstate of the position operator.

by the Baker-Campbell-Hausdorf formula. Expanding for small ξ then gives the above generators. The super covariant derivative generators follow from considering right instead of left multiplication.

One way to read the formula in Eq. (1.129) is that it takes a specific state in the multiplet and then applies all possible supersymmetry transformations on it, obtained by expanding in the parameters θ and $\bar{\theta}$.

Field content

Having obtained a field which is naturally supersymmetric (transforms linearly under the susy algebra) we would like to know which fields it describes. These can be obtained by expanding in the fermionic coordinates. As explained above, Taylor expansions in fermionic coordinates are finite series. For 4 fermionic variables the resulting series has $2^4 = 16$ terms, 2 for each fermionic variable.

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) = f(x) + \theta\,\zeta(x) + \bar{\theta}\,\bar{\eta}(x) +& \\ \theta\theta\, m(x) + \bar{\theta}\bar{\theta}\, n(x) + \bar{\theta}\sigma^\mu\theta\, v_\mu(x)& \\ \theta\theta\bar{\theta}\,\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\,\psi(x) + \theta\theta\bar{\theta}\bar{\theta}\, d(x)&\end{aligned} \tag{1.131}$$

In deriving this it is useful to recall our convention for index-free notation Eqs. (1.44) and (1.46), as well as the results of exercise 1.2.2. Since the fermionic coordinates transform as spin-$\frac{1}{2}$ particles the states with an odd number of fermions are also naturally spin-$\frac{1}{2}$ fermionic fields. The field content is summarized as

$$\begin{array}{cc} \text{bosons} & \text{fermionic} \\ f(x) & \zeta(x) \\ n(x) & \bar{\eta}(x) \\ m(x) & \psi(x) \\ d(x) & \bar{\lambda}(x) \\ v^\mu(x) & \end{array} \tag{1.132}$$

Note that from the commutation relation the SUSY generators have mass dimension $\frac{1}{2}$. This gives the coordinates θ and $\bar{\theta}$ natural mass dimension $-\frac{1}{2}$, which fixes the relative mass dimensions of the fields in the multiplet. If the mass dimension of the field f is k, the mass dimensions of the other fields in the multiplet are fixed as

$$\begin{array}{cc} f(x) & k \\ \zeta(x),\ \bar{\eta}(x) & k+\frac{1}{2} \\ m(x),\ n(x),\ v^\mu(x) & k+1 \\ \psi(x),\ \bar{\lambda}(x) & k+\frac{3}{2} \\ d(x) & k+2 \end{array} \tag{1.133}$$

The mass dimension of the superfield itself is defined to be the same as that of the field $f(x)$.

Moreover, the fields can be given what is known as an R-charge. This follows because supersymmetry algebra has an inner automorphism: a transformation which leaves the algebra invariant,

$$Q_\alpha \to e^{iqa} Q_\alpha \qquad \overline{Q}_{\dot{\alpha}} \to e^{-iqa} \overline{Q}_{\dot{\alpha}} \tag{1.134}$$

for some angle a. The quantity q here is called the R-charge and is usually normalized to $\frac{1}{2}$; it corresponds to a U(1) R-symmetry of the supersymmetry algebra, as introduced in Section 1.2.3.[7] The super-coordinates have a natural R-charge as well. Under the rotation just given, they transform as

$$\theta_\alpha \to e^{-iqa}\theta_\alpha \qquad \bar{\theta}_{\dot{\alpha}} \to e^{iqa}\bar{\theta}_{\dot{\alpha}} \tag{1.135}$$

By the same argument as for the mass dimension, the fields in the multiplet also get natural relative R-charges. Setting the R-charge of the field f to be a we obtain

$$\begin{array}{cc} m(x) & a+1 \\ \zeta(x),\, \bar{\lambda}(x) & a+\frac{1}{2} \\ f(x),\, v_\mu(x),\, d(x) & a \\ \bar{\eta}(x),\, \psi(x) & a-\frac{1}{2} \\ n(x) & a-1 \end{array} \tag{1.136}$$

The R-charge of the superfield itself is defined to be the same as that of the field $f(x)$.

The supersymmetry transformations of the different fields in the multiplet directly follows from the generators in equation (1.125). For this simply act with the generator $\xi Q + \bar{\xi}\bar{Q}$ on the superfield

$$\delta_\xi \Phi(x,\theta,\bar{\theta}) = \left(\xi Q + \bar{\xi}\bar{Q}\right)\Phi(x,\theta,\bar{\theta}) \tag{1.137}$$

and read of the transformation component-by-component

$$\begin{aligned} \delta_\xi \Phi(x,\theta,\bar{\theta}) = \delta_\xi f(x) + \delta_\xi \theta\, \zeta(x) + \bar{\theta}\delta_\xi\, \bar{\eta}(z) + \\ \theta\theta\, \delta_\xi m(x) + \bar{\theta}\bar{\theta}\, \delta_\xi n(x) + \bar{\theta}\sigma^\mu\theta\, \delta_\xi v_\mu(x) \\ \theta\theta\bar{\theta}\, \delta_\xi \bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\, \delta_\xi \psi(x) + \theta\theta\bar{\theta}\bar{\theta}\, \delta_\xi d(x) \end{aligned} \tag{1.138}$$

There is no need to check the closure of the supersymmetry algebra as transformations on the fields as this is automatic by construction. The most important transformation rule to be found reads schematically,

$$\delta_\xi d(x) \propto \partial_\mu J^\mu \tag{1.139}$$

In other words, the component with the most fermionic coordinates in a generic superfield transforms automatically into a total derivative.

A simple and important further observation is that superfields can be added and multiplied just as you would expect: if Φ, Ψ are superfields, then for example

$$a\Phi + b\Psi \qquad a, b \in \mathbb{C}$$

and

$$\Phi\Psi\,, \qquad\qquad \Psi^n \quad \forall\, n \in \mathbb{N}$$

are superfields too. To state this result more formally: superfields form a ring over the complex numbers.

The previous observation can be combined with equation (1.139) to give an action invariant under supersymmetry transformations:

$$\int \mathrm{d}^4x\ \mathrm{d}^2\theta\ \mathrm{d}^2\bar{\theta}\, f(\Phi) \tag{1.140}$$

[7]Note that the only R-symmetry allowed in $\mathcal{N}=1$ is indeed a U(1).

for any function f of the superfield Φ. In most application f will be polynomial. The fermionic integral isolates the 'd'-type term in the superfield that transforms into a total derivative.

As a representation of the supersymmetry algebra the general superfield introduced here is highly reducible. This follows simply from counting the degrees of freedom: there are 8 bosonic and 8 fermionic complex degrees of freedom. This is much more than appear in the on-shell irreducible representations. Irreducible off-shell representations follow from imposing constraints given schematically as, say,

$$F\left(\Phi(x,\theta,\bar{\theta})\right) = 0 \tag{1.141}$$

This function F should be such that:

- F should (anti-)commute with the supersymmetry algebra, without imposing field equations
- $F = 0$ should not directly imply the field equations
- $F = 0$ should have non-trivial solutions

The first of these guarantees that the resulting field still carries an off-shell representation of the susy algebra. The second and third guarantee that the field remains a physically interesting off-shell field. Two examples of such constraints will occupy us in the next few sections.

1.3.3 Chiral superfields

The first constraint which will be studied in these notes is highly reminiscent of (in fact equivalent to, as it will turn out) Eq. (1.90),

$$\boxed{\overline{D}_{\dot{\alpha}}\Phi(x,\theta,\bar{\theta}) = 0\,, \qquad \text{chiral superfields}\,.} \tag{1.142}$$

The explicit form of the derivative operator can be found in equation (1.126). Fields which satisfy this constraint are called *chiral*. There is also a natural conjugate constraint which leads to *anti-chiral* superfields,

$$D_\alpha \Phi^\dagger = 0\,, \qquad \text{anti-chiral superfields}\,. \tag{1.143}$$

These constraints anti-commute with the supersymmetry generators. Let us focus on chiral superfields for now. The constraint equation (1.142) is a first order differential equation and can in this case simply be solved by a coordinate transformation: define a new coordinate y by

$$y^\mu \equiv x^\mu + i\theta\sigma^\mu\bar{\theta}\,. \tag{1.144}$$

This as well as the fermionic coordinate θ are annihilated by $\overline{D}_{\dot{\alpha}}$,

$$\overline{D}_{\dot{\alpha}} y^\mu = 0 \qquad \overline{D}_{\dot{\alpha}}\theta^\alpha = 0\,, \tag{1.145}$$

while

$$\overline{D}_{\dot{\alpha}}\bar{\theta} \neq 0\,. \tag{1.146}$$

In the new coordinates the superspace derivative $\overline{D}$ is especially simple as

$$\overline{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\,, \tag{1.147}$$

and hence the constraint in equation (1.142) is solved by a superfield

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\,\psi(y) + \theta^2 F(y)\,. \tag{1.148}$$

Note that this field has a field content consisting of two complex scalar fields as well as a Weyl fermion: this is the field content of the multiplet studied in Section 1.3.1: Indeed we have just rediscovered the multiplet of that Section in the superfield formalism. The chiral multiplet can also be written as a field on the extended superspace,

$$\Phi(x,\theta,\bar\theta) = \phi(x+i\theta\sigma\bar\theta) + \sqrt{2}\theta\psi(x+i\theta\sigma\bar\theta) + \theta^2 F(x+i\theta\sigma\bar\theta)\,, \tag{1.149}$$

which by formal Taylor expansion around x becomes

$$\boxed{\begin{aligned}\Phi(x,\theta,\bar\theta) =& \,\phi(x) + \sqrt{2}\theta\,\psi(x) + \theta^2\,F(x)\\ &+ i\theta\sigma^\mu\bar\theta\,\partial_\mu\phi(x) - \frac{i}{\sqrt{2}}\theta^2\,\partial_\mu\psi(x)\sigma^\mu\bar\theta + \frac{1}{4}\theta^2\bar\theta^2\,\partial_\mu\partial^\mu\,\phi(x)\,.\end{aligned}} \tag{1.150}$$

Anti-chiral superfields can be dealt with in a completely equivalent way, noticing that the supercovariant derivative D_α annihilates $\bar\theta$ as well as the shifted coordinate $\bar y = x - i\theta\sigma\bar\theta$. Therefore the most general solution to the constraint Eq. (1.143) is

$$\Phi^\dagger(x,\theta,\bar\theta) = \phi^*(\bar y) + \sqrt{2}\bar\theta\,\overline{\psi}(\bar y) + \bar\theta^2\,\overline{F}(\bar y)\,, \tag{1.151}$$

which has an expansion

$$\begin{aligned}\Phi^\dagger(x,\theta,\bar\theta) =& \,\phi^*(x) + \sqrt{2}\bar\theta\,\overline{\psi}(x) + \bar\theta^2\,\overline{F}(x)\\ &- i\theta\sigma^\mu\bar\theta\,\partial_\mu\phi^*(x) + \frac{i}{\sqrt{2}}\bar\theta^2\theta\,\sigma^\mu\partial_\mu\overline{\psi}(x) + \frac{1}{4}\theta^2\bar\theta^2\,\partial_\mu\partial^\mu\,\phi^*(x)\,.\end{aligned} \tag{1.152}$$

Another way to solve the chiral constraint is to note that:

$$\overline{D}_{\dot\alpha}\overline{D}_{\dot\beta}\overline{D}_{\dot\gamma} = 0 \tag{1.153}$$

because the derivative anti-commute. Hence

$$\Phi = \overline{D}\,\overline{D}\,U \tag{1.154}$$

is a chiral superfield for a generic, unconstrained superfield U. This is seen to lead to the same result as above in equation (1.148) on the $y,\theta,\bar\theta$ space where the superspace derivative $\overline{D}$ is simple, see equation (1.147).

The supersymmetry transformations of the components of the chiral superfield can be obtained as before. Actually, from writing the superfield as in equation (1.154) and with the superspace derivative as in equation (1.147) it follows that $F(y)$ transforms as $d(y - i\theta\sigma^\mu\bar\theta)$, i.e. as a total derivative. Hence

$$\boxed{\mathcal{L} = \int \mathrm{d}^4y\;\mathrm{d}^2\theta\;\Phi(y,\theta)} \tag{1.155}$$

is a supersymmetry invariant term which can be added to a Lagrangian for a generic chiral superfield Φ. Note

$$\int \mathrm{d}^4y\;\mathrm{d}^2\theta\,\Phi(y,\theta) = \int \mathrm{d}^4x\;\mathrm{d}^2\theta\;\Phi(x+i\theta\sigma^\mu\bar\theta,\theta)\,. \tag{1.156}$$

Similarly,

$$\int \mathrm{d}^4\bar{y}\ \mathrm{d}^2\bar{\theta}\,\Phi^\dagger(\bar{y},\bar{\theta}) \tag{1.157}$$

is invariant under supersymmetry transformations for a generic anti-chiral superfield $\Phi^\dagger$.

As before, chiral superfields can be added and multiplied: the result is still a chiral superfield. This is a direct consequence of the fact that the constraint involves a linear derivative which obeys the (super)-Leibniz rule. The same holds for anti-chiral superfields. However, the product of a chiral and an anti-chiral superfield is generically *not* chiral or anti-chiral: it is a general superfield.

A special field which is both anti-chiral as well as chiral is a constant. Finally, we note that the full superspace integral over a chiral or antichiral superfield is a total derivative:

$$\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\ \Phi = \partial_\mu\partial^\mu\,\phi(x)\,, \qquad \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\ \Phi^\dagger = \partial_\mu\partial^\mu\,\phi^*(x)\,, \tag{1.158}$$

since the superspace integral projects on the $\theta^2\bar{\theta}^2$ component.

Exercise 9: $\mathcal{N}=2$ superspace

Let's explore an extended superspace by introducing more fermionic coordinates: the set of coordinates is now $x^\mu, \theta^I_\alpha, \bar{\theta}^I_{\dot{\alpha}}$ where I runs from 1 to 2.

a. Construct supersymmetry generators Q^I_α, and $\bar{Q}^I_{\dot{\alpha}}$ as well as D^I_α and $\bar{D}^I_{\dot{\alpha}}$

b. How many components are in the unconstrained superfield

$$\phi(x^\mu, \theta^I_\alpha, \bar{\theta}^I_{\dot{\alpha}})$$

c. What is the highest spin field that appears in this multiplet?

d. Set $D^I_\alpha\phi = 0$ for $I = 1, 2$. How many fields are now in the multiplet? Which types of field (representations of the Lorentz group) appear?

1.3.4 Wess-Zumino models

By a *Wess-Zumino model* we denote a renormalizable supersymmetric field theory for chiral and antichiral superfields. Since we have now assembled all the superspace tools to construct such a theory, writing down a sensible Lagrangian is fairly straightforward. Let Φ_i be a collection of chiral superfields, and $\Phi^\dagger_i$ their conjugate antichiral superfields,

$$\overline{D}_{\dot{\alpha}}\Phi_i = D_\alpha\Phi^\dagger_i = 0\,. \tag{1.159}$$

From the previous two sections we have learned that the superspace integral of any superfield is SUSY-invariant. Since products and sums of superfields are again superfields, the term

$$\mathcal{L}_K = \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\ K\left(\Phi_i, \Phi^\dagger_i\right) \tag{1.160}$$

is real and SUSY-invariant for any real function K. Furthermore, we have learned that the "half-superspace integral" $\int \mathrm{d}^2\theta$ of any chiral superfield is SUSY-invariant, as is the $\int \mathrm{d}^2\bar\theta$ integral of any antichiral superfield. Since products and sums of (anti-)chiral superfields are again (anti-)chiral, the terms

$$\mathcal{L}_W = \int \mathrm{d}^2\theta\, W(\Phi_i) + \int \mathrm{d}^2\bar\theta\, W^\dagger(\Phi_i^\dagger) \tag{1.161}$$

are real and SUSY-invariant for any holomorphic function W. While there exist other SUSY-invariant expressions besides the terms in Eq. (1.160) and Eq. (1.161), these involve explicit supercovariant derivatives and are consequently of higher order in space-time derivatives. By power counting, we can determine what terms K and W can at most contain to yield a renormalizable field theory.

The function K is called the *Kähler potential*. It must have mass dimension 2 since the superspace integration measure has dimension 2. Therefore its most general form is, if only positive powers of mass parameters are allowed to ensure renormalizability,

$$K = K_0 + K_i\,\Phi_i + K_i^\dagger \Phi_i^\dagger + Y_{ij}\Phi_i\Phi_j + Y_{ij}^\dagger\Phi_i^\dagger\Phi_j^\dagger + K_{ij}\Phi_i^\dagger\Phi_j\,, \tag{1.162}$$

with some c-number parameters K_0, K_i, Y_{ij} and K_{ij}. However, K_0 is irrelevant since it vanishes under the integral Eq. (1.160), and the K_i and Y_{ij} are irrelevant since the same superspace integral over an (anti-)chiral superfield is a total derivative. Thus, without loss of generality,

$$K = K_{ij}\Phi_i^\dagger\Phi_j\,, \tag{1.163}$$

where K_{ij} is some dimensionless matrix. With the help of Eqs. (1.150) and (1.152) it is straightforward to figure out the $\theta^2\bar\theta^2$ component of $\Phi_i^\dagger\Phi_j$ (→ Exercise), from which we obtain the following component representation of Eq. (1.160):

$$\begin{aligned}
&\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\; K_{ij}\,\Phi_i\Phi_j^\dagger \\
&\quad = K_{ij}\left(\frac{1}{4}\phi_i^*\partial^2\phi_j + \frac{1}{4}\left(\partial^2\phi_i^*\right)\phi_j - \frac{1}{2}\partial_\mu\phi_i^*\partial^\mu\phi_j + \frac{i}{2}\partial_\mu\overline\psi_i\overline\sigma^\mu\psi_j - \frac{i}{2}\overline\psi_i\overline\sigma^\mu\partial_\mu\psi_j + F_i^*F_j\right) \\
&\quad = K_{ij}\left(-\partial_\mu\phi_i^*\partial^\mu\phi_j - i\overline\psi_i\overline\sigma^\mu\partial_\mu\psi_j + F_i^*F_j\right) + \text{total derivatives.}
\end{aligned} \tag{1.164}$$

The first two terms are well-behaved kinetic energy terms for the fields ϕ_i and ψ_i, provided we choose K_{ij} real, symmetric and positive definite. With this choice we can redefine our fields to bring K_{ij} into the form $K_{ij} = \delta_{ij}$, thus obtaining standard kinetic terms for canonically normalized component fields. In summary, the superspace expression

$$\boxed{\mathcal{L}_K = \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\; \Phi_i^\dagger\Phi_i} \tag{1.165}$$

gives rise to canonical kinetic terms for the propagating fields ϕ_i and ψ_i, as well as a term $F_i^*F_i$ for the F-type auxiliary fields.

The function W is called the *superpotential*. It must have dimension 3 because the $d^2\theta$ integration has dimension 1. Again allowing only for positive powers of dimensionful parameters for renormalizability, we arrive at the expansion

$$\boxed{W(\Phi_i) = f_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k\,.} \tag{1.166}$$

We could have added an irrelevant constant (which drops out under the $d^2\theta$ integral). The superpotential parameters are some dimension-two parameters f_i, a dimension-one symmetric matrix m_{ij} which will turn out to be a mass matrix, and some symmetric dimensionless couplings g_{ijk}. In components, the Lagrangian Eq. (1.161) then becomes

$$\begin{aligned}
&\int \mathrm{d}^2\theta\, W(\Phi_i) \\
&= f_i\, F_i + \frac{m_{ij}}{2}\left(\phi_i F_j + F_i \phi_j - \psi_i \psi_j\right) \\
&\quad + \frac{g_{ijk}}{3}\left(F_i \phi_j \phi_k + \phi_i F_j \phi_k + \phi_i \phi_j F_k - \phi_i \psi_j \psi_k - \psi_i \phi_j \psi_k - \psi_i \psi_j \phi_k\right) .
\end{aligned} \tag{1.167}$$

As we did in the simple single-field example in Section 1.3.1, we can eliminate the auxiliary fields F_i using their algebraic equations of motion. Writing the complete Lagrangian as

$$\mathcal{L} = \mathcal{L}_K + \mathcal{L}_W = \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, K + \int \mathrm{d}^2\theta\, W + \int \mathrm{d}^2\bar{\theta}\, W^\dagger \tag{1.168}$$

with $K = \Phi_i^\dagger \Phi_i$ and W as in Eq. (1.166), these read

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial F_i} = F_i^* + f_i + m_{ij}\phi_j + g_{ijk}\phi_j\phi_k\,, \\
0 &= \frac{\partial \mathcal{L}}{\partial F_i^*} = F_i + f_i^* + m_{ij}^*\phi_j^* + g_{ijk}^*\phi_j^*\phi_k^*\,.
\end{aligned} \tag{1.169}$$

Solving for F_i and F_i^*, and substituting the resulting expressions back into $\mathcal{L}$, we arrive at

$$\begin{aligned}
\mathcal{L} &= -\partial_\mu \phi_i^* \partial^\mu \phi_i - i\overline{\psi}_i \overline{\sigma}^\mu \partial_\mu \psi_i \\
&\quad - \left(\frac{m_{ij}}{2}\psi_i\psi_j + \text{h.c.}\right) - \frac{g_{ijk}}{3}\left(\phi_i\psi_j\psi_k + \psi_i\phi_j\psi_k + \psi_i\psi_j\phi_k + \text{h.c.}\right) \\
&\quad - \left(f_i + m_{ij}\phi_i + g_{ijk}\phi_j\phi_k\right)\left(f_i^* + m_{il}^*\phi_l^* + g_{ilm}^*\phi_l^*\phi_m^*\right) .
\end{aligned} \tag{1.170}$$

Here and in the following "+ h.c." denotes the hermitian conjugate of the preceding expression.

The Lagrangian Eq. (1.170) is the most general renormalizable supersymmetric component Lagrangian for chiral and antichiral multiplets. Note that it contains all the couplings expected in a renormalizable QFT of scalar and spinor fields: scalar tadpoles, mass terms for both the fermions and the scalars, scalar cubic and quartic self-interactions, and Yukawa couplings. However, supersymmetry imposes relations between these couplings (notably between the quartic interactions and the Yukawa couplings, as well as between the spinor and scalar masses) which strongly restrict the form $\mathcal{L}$ can take. A particularly remarkable feature is that the scalar potential

$$\mathcal{V} = \sum_i \left|\frac{\partial W}{\partial \Phi_i}\right|^2 \Bigg|_{\Phi_i = \phi_i} = \sum_i \left|f_i + m_{ij}\phi_i + g_{ijk}\phi_j\phi_k\right|^2 \tag{1.171}$$

is positive definite. We will explore the meaning of this, and the connection with spontaneous supersymmetry breaking, later on in the lectures.

Exercise 10: Wess-Zumino Lagrangian: kinetic terms

Φ and $\Phi^\dagger$ are a chiral and an anti-chiral superfield respectively.

a. Show
$$\int \mathrm{d}^4x\ \mathrm{d}^2\theta\ \mathrm{d}^2\bar{\theta}\,\Phi^2 = \int \mathrm{d}^4x\ \mathrm{d}^2\theta\ \mathrm{d}^2\bar{\theta}\,(\Phi^\dagger)^2 = 0$$
.

b. Obtain the kinetic term of the Wess-Zumino Lagrangian by computing
$$\int \mathrm{d}^4x\ \mathrm{d}^2\theta\ \mathrm{d}^2\bar{\theta}\Phi^\dagger\Phi\,.$$

c. Show
$$\frac{1}{2}\int \mathrm{d}^4x\ \mathrm{d}^2\theta\, m\Phi^2 = m\,\phi(x)F(x) + m\,\psi(x)\psi(x)\,.$$

d. Show by 'integrating out' the auxiliary fields (i.e. solving their field equations) that the Lagrangian obtained by adding the terms under a) and b) has a canonical mass term for the scalar field.

1.3.5 Vector superfields and supersymmetric gauge theories

Having seen how to construct a field theory from chiral superfields, which contain scalar and spinor components, we are now ready to introduce another important superfield. In order to describe gauge fields, we also need component fields which transform as vectors. We begin with the simplest case of a U(1) gauge field.

Vector superfields and U(1) gauge fields

A vector is contained in the *vector superfield* or *real superfield*, along with spinor and scalar degrees of freedom. It is defined by the constraint
$$V = V^\dagger\,. \tag{1.172}$$
Its component expansion reads
$$\begin{aligned} V(x,\theta,\bar{\theta}) = {} & C(x) - \sqrt{2}i\,\theta\chi(x) + \sqrt{2}i\,\bar{\theta}\overline{\chi}(x) - i\theta^2\,G(x) + i\bar{\theta}^2\,G^*(x) \\ & - \theta\sigma^\mu\bar{\theta}A_\mu(x) + i\,\theta^2\bar{\theta}\left(\bar{\lambda}(x) - \frac{i}{\sqrt{2}}\bar{\sigma}^\mu\partial_\mu\chi(x)\right) - i\bar{\theta}^2\theta\left(\lambda(x) - \frac{i}{\sqrt{2}}\sigma^\mu\partial_\mu\overline{\chi}(x)\right) \\ & + \frac{1}{2}\theta^2\bar{\theta}^2\left(D(x) + \frac{1}{2}\partial_\mu\partial^\mu\,C(x)\right)\,. \end{aligned} \tag{1.173}$$

Here $C(x)$ and $D(x)$ are real scalar fields, $A_\mu(x)$ is a real vector field, $G(x)$ is a complex scalar, and λ and χ are spinors. The peculiar combinations appearing in the $\theta^2\bar{\theta}$, $\bar{\theta}^2\theta$ and $\theta^2\bar{\theta}^2$ terms are chosen with some hindsight (we just define the fields λ and D accordingly). To see why this particular combination is useful, consider a chiral superfield X defined by
$$X(x,\theta,\bar{\theta}) = \frac{1}{2}\left[\alpha(y) + i\,C(y)\right] + \sqrt{2}\theta\chi(y) + \theta^2\,G(y) \tag{1.174}$$

where α is some real function and $y \equiv x + i\theta\sigma\bar\theta$ as in the previous chapter. According to Eqs. (1.150) and (1.152), the component expansion of the superfield $iX + (iX)^\dagger$ reads

$$\begin{aligned} iX - iX^\dagger = & - C(x) + \sqrt{2}i\,\theta\chi(x) - \sqrt{2}i\bar\theta\overline{\chi}(x) + iG(x)\theta^2 - iG^*(x)\bar\theta^2 \\ & - \theta\sigma^\mu\bar\sigma\partial_\mu\alpha(x) + \frac{1}{\sqrt{2}}\theta^2\partial_\mu\chi(x)\sigma^\mu\bar\theta - \frac{1}{\sqrt{2}}\bar\theta^2\partial_\mu\overline{\chi}(x)\sigma^\mu\theta \\ & - \frac{1}{4}\theta^2\bar\theta^2\partial_\mu\partial^\mu\, C(x)\,. \end{aligned} \tag{1.175}$$

In a theory which is invariant under *super-gauge transformations*

$$V \;\to\; V + i\left(\Lambda - \Lambda^\dagger\right) \tag{1.176}$$

for arbitrary chiral superfields Λ, we can fix the gauge by choosing $\Lambda = X$. This sends

$$\begin{aligned} &C \to 0\,, \qquad \chi \to 0\,, \qquad G \to 0\,, \\ &A_\mu \to A'_\mu = A_\mu + \partial_\mu\alpha\,, \qquad \lambda \to \lambda\,, \qquad D \to D\,. \end{aligned} \tag{1.177}$$

such that the superfield V takes the simple form

$$V = -\theta\sigma^\mu\bar\theta\, A'_\mu(x) + i\,\theta^2\bar\theta\,\bar\lambda(x) - i\,\bar\theta^2\theta\lambda(x) + \frac{1}{2}\theta^2\bar\theta^2\,D(x)\,. \tag{1.178}$$

The fields C, χ and G can be completely gauged away. This choice of gauge is called *Wess-Zumino gauge*. Since $\alpha(x)$ is arbitrary, the Wess-Zumino gauge does not fix the gauge for the ordinary U(1) transformation

$$A_\mu \;\to\; A_\mu + \partial_\mu\alpha\,. \tag{1.179}$$

Stated differently, in a theory with the symmetry under the transformation of Eq. (1.176), we can go to Wess-Zumino gauge and the transformation Eq. (1.179) will remain a symmetry.

An important property of Wess-Zumino gauge is that all powers higher than quadratic in V vanish,

$$V^2 = -\frac{1}{2}\theta^2\bar\theta^2\,A_\mu A^\mu\,, \qquad V^3 = V^4 = \ldots = 0 \tag{1.180}$$

which in particular implies that e^V has a finite power series expansion; we will come back to this shortly.

In order to construct a sensible field theory from V, we need kinetic terms for at least part of its component fields. We introduce the *field strength superfield* W_α by

$$W_\alpha = -\frac{1}{4}\overline{DD}D_\alpha V\,. \tag{1.181}$$

Note the "external" spinor index: W_α contains two superfields W_1 and W_2 which together transform as a left-handed spinor under Lorentz transformations; W_1 and W_2 themselves are superfields which contain other components according to the θ-expansion. They are in fact chiral superfields since

$$\overline{D}_{\dot\beta}W_\alpha = \overline{D}_{\dot\beta}\overline{DD}D_\alpha V = 0 \tag{1.182}$$

(remember that powers of $\overline{D}$ higher than two vanish). They are also invariant under supergauge transformations:

$$\begin{aligned} W_\alpha &\to W_\alpha - \frac{1}{4}\overline{DD}D_\alpha\left(i\Lambda - i\Lambda^\dagger\right) = W_\alpha - \frac{i}{4}\overline{DD}D_\alpha\Lambda \\ &= W_\alpha - \frac{i}{4}\overline{D}(\overline{D}D_\alpha + D_\alpha\overline{D})\Lambda = W_\alpha + \frac{i}{4}\overline{D}^{\dot\alpha}\{\overline{D}_{\dot\alpha}, D_\alpha\}\Lambda \\ &= W_\alpha + \frac{1}{2}\overline{D}^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\Lambda = W_\alpha + \frac{1}{2}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\overline{D}^{\dot\alpha}\Lambda = W_\alpha\,, \end{aligned} \tag{1.183}$$

where we have used that Λ is chiral and $\Lambda^\dagger$ is antichiral.

The component expansion of W_α in Wess-Zumino gauge reads, in terms of $y \equiv x + i\theta\sigma\bar\theta$,

$$\begin{aligned} W_\alpha(x,\theta,\bar\theta) =&\; - i\lambda_\alpha(y) + D(y)\theta_\alpha - \frac{i}{2}\sigma^\mu_{\alpha\dot\alpha}\overline{\sigma}^{\nu\dot\alpha\beta}\theta_\beta\,\left(\partial_\mu A_\nu(y) - \partial_\nu A_\mu(y)\right) + \theta^2\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar\lambda^{\dot\alpha}(y) \\ =&\; - i\lambda_\alpha(y) + D(y)\theta_\alpha - \frac{i}{2}\sigma^\mu_{\alpha\dot\alpha}\overline{\sigma}^{\nu\dot\alpha\beta}\theta_\beta\,F_{\mu\nu}(y) + \theta^2\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar\lambda^{\dot\alpha}(y)\,. \end{aligned} \tag{1.184}$$

The complete θ expansion follows as for the general chiral superfield. Note that W_α contains the field strength $F_{\mu\nu}$ as one of its components, as well as derivatives acting on λ. It is therefore a good starting point to construct a kinetic Lagrangian for V. To do so, we write down the θ^2 component of $W^\alpha W_\alpha$,

$$\int \mathrm{d}^2\theta\; W^\alpha W_\alpha = -2i\,\lambda(x)\sigma^\mu\partial_\mu\bar\lambda(x) - \frac{1}{2}F^{\mu\nu}(x)F_{\mu\nu}(x) + \frac{i}{4}\epsilon_{\mu\nu\kappa\lambda}F^{\mu\nu}(x)F^{\kappa\lambda}(x) + D(x)^2\,. \tag{1.185}$$

Since W_α is chiral, this transforms into a space-time derivative. A kinetic Lagrangian is then obtained by adding the conjugate expression,

$$\begin{aligned} \mathcal{L}_{\text{gauge-kin.}} &= \frac{1}{4}\int \mathrm{d}^2\theta\; W^\alpha W_\alpha + \frac{1}{4}\int \mathrm{d}^2\bar\theta\; \overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha} \\ &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\bar\lambda + \frac{1}{2}D^2 + (\text{total derivatives})\,. \end{aligned} \tag{1.186}$$

It is also possible to obtain this Lagrangian as an integral over the full superspace, rather than just the chiral and antichiral halves. To do so, note that

$$W^\alpha W_\alpha = -\frac{1}{4}\overline{DD}W^\alpha D_\alpha V \tag{1.187}$$

whereby

$$\mathcal{L}_{\text{gauge-kin.}} = \frac{1}{4}\int \mathrm{d}^2\theta\;\mathrm{d}^2\bar\theta\;\left(W^\alpha D_\alpha V + \overline{W}_{\dot\alpha}\overline{D}^{\dot\alpha}V\right) + (\text{total derivatives})\,. \tag{1.188}$$

From the Lagrangian Eq. (1.186) it is evident that the vector supermultiplet contains a gauge field and a spinor (the *gaugino*) as propagating degrees of freedom. The field $D(x)$ is an auxiliary field, much like the field $F(x)$ of the chiral supermultiplet. At the level of a non-interacting theory as defined by Eq. (1.186), it can be trivially integrated out by its equation of motion, which reads

$$D = 0\,. \tag{1.189}$$

Matter couplings

The logical next step is to couple the gauge field we found to charged matter. Candidate matter fields are contained in chiral supermultiplets. Under a global U(1) symmetry which commutes with supersymmetry, a chiral superfield $\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y)$ should transform as

$$\Phi \;\rightarrow\; e^{-2iq\,\alpha}\Phi \tag{1.190}$$

where the q is a U(1) charge, and α is a constant. We cannot just promote $\alpha \rightarrow \alpha(x)$ to obtain a U(1) gauge symmetry, since this would break supersymmetry. Instead we have to promote α to a full superfield. The proper generalization of Eq. (1.190) reads

$$\Phi \;\rightarrow\; e^{-2iq\,\Lambda(x,\theta,\bar{\theta})}\Phi \tag{1.191}$$

where Λ is a chiral superfield, $\overline{D}_{\dot{\alpha}}\Lambda = 0$. Anti-chiral superfields transform as

$$\Phi^\dagger \;\rightarrow\; e^{2iq\,\Lambda^\dagger(x,\theta,\bar{\theta})}\Phi^\dagger\,. \tag{1.192}$$

This makes sense because products of chiral superfields are again chiral superfields.

In the previous chapter we saw that the kinetic action for a chiral superfield is obtained as a superspace integral over $\Phi^\dagger\Phi$. This is not gauge invariant:

$$\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\;\Phi^\dagger\Phi \;\rightarrow\; \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\;\Phi^\dagger e^{-2iq(\Lambda-\Lambda^\dagger)}\Phi\,. \tag{1.193}$$

We can compensate for the gauge non-invariance by introducing a gauge field into the kinetic Lagrangian. Recalling Eq. (1.176), the combination

$$\mathcal{L}_{\mathrm{kin.}} = \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\;\Phi^\dagger e^{2q\,V}\Phi \tag{1.194}$$

is gauge invariant. It is also renormalizable, because we can go to Wess-Zumino gauge where the e^V power series terminates after just a few terms. This is seen by writing out the component form of $\mathcal{L}_{\mathrm{kin.}}$ in Wess-Zumino gauge:

$$\begin{aligned}\mathcal{L}_{\mathrm{kin.}} =& - D_\mu\phi(D^\mu\phi)^* - i\overline{\psi}\overline{\sigma}^\mu D_\mu\psi + |F|^2 \\ &- \sqrt{2}i\,q\left(\phi\overline{\lambda\psi} - \phi^*\lambda\psi\right) + qD|\phi|^2 \;+\; (\text{total derivatives})\,.\end{aligned} \tag{1.195}$$

Here $D_\mu = \partial_\mu + iq\,A_\mu$ is the usual gauge-covariant derivative (not to be confused with the auxiliary field of the vector multiplet $D = D(x)$ which appears in the second line, or the supercovariant derivative D_α).

In the previous chapter we saw that supersymmetric interactions between multiple chiral superfields can be written in terms of chiral superspace integrals. Renormalizable interactions are described by a Lagrangian

$$\mathcal{L}_{\mathrm{int.}} = \int \mathrm{d}^2\theta\left(f_i\,\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k\right) + \text{h.c.} \tag{1.196}$$

This is gauge invariant upon assigning U(1) charges q_i to the chiral superfields Φ_i, provided that

$$\begin{aligned} f_i &= 0 \qquad \text{if} \quad q_i \neq 0\,, \\ m_{ij} &= 0 \qquad \text{if} \quad q_i + q_j \neq 0\,, \\ g_{ijk} &= 0 \qquad \text{if} \quad q_i + q_j + q_k \neq 0\,. \end{aligned} \tag{1.197}$$

We can now write down "super-QED": a supersymmetric theory of a massive Dirac spinor (the "electron") coupled to a U(1) gauge field. We need two chiral superfields Φ and $\widetilde{\Phi}$ of charges $\pm e$, housing the two Weyl spinors needed to make up a Dirac spinor, and a vector superfield V. The Lagrangian is

$$\mathcal{L}_{\rm SQED} = \int \mathrm{d}^4\theta \left(\Phi^\dagger e^{2eV}\Phi + \widetilde{\Phi}^\dagger e^{-2eV}\widetilde{\Phi}\right) + \int \mathrm{d}^2\theta\, m\Phi\widetilde{\Phi} + \text{h.c.} + \frac{1}{4}\int \mathrm{d}^2\theta\, W^\alpha W_\alpha + \text{h.c.} \tag{1.198}$$

or in components

$$\begin{aligned}\mathcal{L}_{\rm SQED} = & -D_\mu\phi(D^\mu\phi)^* - D_\mu\widetilde{\phi}(D^\mu\widetilde{\phi})^* - i\overline{\psi}\overline{\sigma}^\mu D_\mu\psi - i\overline{\widetilde{\psi}}\overline{\sigma}^\mu D_\mu\widetilde{\psi} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\bar{\lambda} \\ & -\sqrt{2}i\, e\left(\phi\overline{\psi\lambda} - \phi^*\psi\lambda - \widetilde{\phi}\overline{\widetilde{\psi}\lambda} + \widetilde{\phi}^*\widetilde{\psi}\lambda\right) \\ & -m\left(\psi\widetilde{\psi} + \overline{\psi\widetilde{\psi}}\right) - \mathcal{V}\left(\phi,\phi^*,\widetilde{\phi},\widetilde{\phi}^*,F,F^*,\widetilde{F},\widetilde{F}^*,D\right)\end{aligned} \tag{1.199}$$

with the scalar potential

$$\mathcal{V} = -|F|^2 - |\widetilde{F}|^2 - m\left(\phi\widetilde{F} + \widetilde{\phi}F + \phi^*\widetilde{F}^* + \widetilde{\phi}^*F^*\right) - \frac{1}{2}D^2 - eD\left(|\phi|^2 - |\widetilde{\phi}|^2\right) . \tag{1.200}$$

By their equations of motion, we can integrate out the scalar auxiliary fields F, $\widetilde{F}$ and D:

$$\begin{aligned} 0 = \frac{\partial\mathcal{L}}{\partial F} &= F^* + m\,\widetilde{\phi} && \Rightarrow \quad F^* = -m\,\widetilde{\phi}\,, \quad F = -m\,\widetilde{\phi}^*\,, \\ 0 = \frac{\partial\mathcal{L}}{\partial \widetilde{F}} &= \widetilde{F}^* + m\,\phi && \Rightarrow \quad \widetilde{F}^* = -m\,\phi\,, \quad \widetilde{F} = -m\,\phi^*\,, \\ 0 = \frac{\partial\mathcal{L}}{\partial D} &= D + e\left(|\phi|^2 - |\widetilde{\phi}|^2\right) && \Rightarrow \quad D = -e\left(|\phi|^2 - |\widetilde{\phi}|^2\right) . \end{aligned} \tag{1.201}$$

Re-substituting into Eq. (1.200) yields

$$\mathcal{V} = m^2|\phi|^2 + m^2|\widetilde{\phi}|^2 + \frac{e^2}{2}\left(|\phi|^2 - |\widetilde{\phi}|^2\right)^2 . \tag{1.202}$$

From Eqns. (1.199) and (1.202) we see that supersymmmetric QED contains not only the electron and the photon but also the other propagating states in their respective supermultiplets: We have to complex scalar electron partners ("selectrons") ϕ and $\widetilde{\phi}$, as well as a spinorial photon partner (the "photino") λ. While their interactions in Eq. (1.199) look quite complicated at first sight, they are in fact strongly restricted by supersymmetry. Note, for instance, that specifying the gauge coupling completely fixes the interaction terms: all Yukawa couplings in the second line of Eq. (1.199) as well the quartic scalar coupling in the scalar potential are essentially given by e and e^2 respectively. Furthermore, the masses of ϕ, of $\widetilde{\phi}$, and the Dirac mass of the electron are all given by the single parameter m.

Non-abelian generalization

Eventually we also want to describe non-abelian gauge fields. For this our superfields should take values in some vector space carrying a representation of some compact Lie algebra. The transformation law Eq. (1.192) becomes

$$\Phi_A \to \left(e^{-i\Lambda}\right)_{AB} \Phi_B \tag{1.203}$$

where $\Lambda_{AB} = \Lambda^a T^a_{AB}$, and the T^a_{AB} are hermitian generators of the gauge group G in some representation.

The transformation law for the gauge field, given by Eq. (1.176) in the abelian case, becomes

$$e^V \to e^{V'} = e^{-i\Lambda^\dagger} e^V e^{i\Lambda} . \tag{1.204}$$

Here V is also Lie algebra-valued, $V_{AB} = V^a T^a_{AB}$. With this definition, the kinetic Lagrangian for a charged chiral superfield

$$\mathcal{L}_{\text{kin.}} = \int \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \, \Phi^\dagger e^V \Phi \tag{1.205}$$

is gauge invariant.

Eq. (1.204) cannot be easily converted into a closed-form expression for the transformation law for V, because Λ and V do not commute. However, expanding both sides up to linear order we have

$$\begin{aligned} e^{V'} &= 1 + V' + \ldots = (1 - i\Lambda^\dagger + \ldots)(1 + V + \ldots)(1 + i\Lambda + \ldots) \\ &= 1 + V + i(\Lambda - \Lambda^\dagger) + \ldots \end{aligned} \tag{1.206}$$

This shows that we can go to a Wess-Zumino gauge in non-abelian theories, just as in the U(1) case, in which $V^3 = 0$ and the power series expansion of e^V is finite.

Finally, the generalization of the field strength superfield to non-abelian theories is

$$W_\alpha = -\frac{1}{4}\overline{DD} e^{-V} D_\alpha e^V , \tag{1.207}$$

with $V = V^A T^A{}_{BC}$ and the generators $T^A{}_{BC}$ in the adjoint representation, $T^A{}_{BC} = f^A{}_{BC}$. The field strength is no longer gauge invariant, but transforms homogeneously:

$$\begin{aligned} W_\alpha \to W'_\alpha = &-\frac{1}{4}\overline{DD}\left(e^{-i\Lambda}e^{-V}e^{i\Lambda^\dagger}D_\alpha e^{-i\Lambda^\dagger}e^V e^{i\Lambda}\right) \\ = &-\frac{1}{4}\overline{DD}\left(e^{-i\Lambda}e^{-V}(D_\alpha e^V)e^{i\Lambda} + e^{-i\Lambda}D_\alpha e^{i\Lambda}\right) \\ &= e^{-i\Lambda}W_\alpha e^{i\Lambda} - \frac{1}{4}e^{-i\Lambda}\overline{DD}D_\alpha e^{i\Lambda} \\ &= e^{-i\Lambda}W_\alpha e^{i\Lambda} - \frac{1}{4}e^{-i\Lambda}\overline{D}\{\overline{D}, D_\alpha\}e^{i\Lambda} \\ &= e^{-i\Lambda}W_\alpha e^{i\Lambda} . \end{aligned} \tag{1.208}$$

Here we have repeatedly used $\overline{D}e^{i\Lambda} = 0$ and $D_\alpha e^{-i\Lambda^\dagger} = 0$ (because Λ is chiral and $\Lambda^\dagger$ is antichiral), similar as in Eqns. (1.183). A kinetic term for the gauge field is therefore given by

$$\mathcal{L}_{\text{gauge}-\text{kin.}} = \frac{1}{2\,g^2}\int \mathrm{d}^2\theta \; \mathrm{tr}\, W^\alpha W_\alpha + \text{h.c.} \tag{1.209}$$

where g is the gauge coupling constant.

Exercise 11: Supersymmetry transformations

Find the supersymmetry variations $\delta_\xi F^{\mu\nu}$, $\delta_\xi \lambda$, and $\delta_\xi D$ for the gauge-invariant components of a U(1) vector supermultiplet.

Exercise 12: Kinetic Lagrangians for gauge and matter fields

a. Verify Eq. (1.187), and use it to prove that the kinetic Lagrangian of Eq. (1.188) is equivalent to that of Eq. (1.186).

b. Verify Eq. (1.195).

1.3.6 The general $\mathcal{N} = 1$, $d = 4$ SUSY Lagrangian

To be completed

1.3.7 Spontaneous supersymmetry breaking

To be completed

Chapter 2

Global supersymmetry: Applications

2.1 Supersymmetry as a tool to understand non-perturbative gauge dynamics

In this Section we will get a glimpse on how supersymmetry may allow us to make exact statements about supersymmetric gauge theories, even in a regime where perturbation theory does not apply. The crucial point is that some functions defining a supersymmetric theory are *holomorphic* functions of the chiral superfields, and even of the coupling constants as will become clear. From complex analysis we know that holomorphy is a very stringent constraint; exploiting this constraint will allow us to establish some interesting properties of supersymmetric systems. We start by exploring the holomorphicity of the superpotential in a simpler theory, the Wess-Zumino model of chiral superfields.

2.1.1 Holomorphy and non-renormalization in Wess-Zumino models

By a general Wess-Zumino model we mean any model of interacting chiral superfields. We have seen that the most general two-derivative Lagrangian can be written as

$$\mathcal{L} = \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\; K(\Phi_i^\dagger, \Phi_i) + \int \mathrm{d}^2\theta\; W(\Phi_i) + \text{h.c.} \tag{2.1}$$

where K is a real function called the *Kähler potential* and W is a holomorphic function called the *superpotential*.

Suppose that we have a theory defined somewhere at high energies in the ultraviolet. By successively integrating over short-wavelength fluctuations, up to some scale μ, we obtain a *Wilsonian effective action* which is valid for processes at energies below μ. Its couplings will depend on the scale μ. It is a standard principle of quantum field theory that, at least in a generic non-supersymmetric setting, quantum fluctuations will generate all possible terms in the effective action which are compatible with the symmetries of the theory.

The situtation is very different in supersymmetry. Since the theory is supersymmetric, one should be able to write also the (two-derivative part of the) Wilsonian effective action using a Lagrangian of the form (2.1). One may therefore expect that one must replace $K \to K_{\text{eff}}(\mu)$ and $W \to W_{\text{eff}}(\mu)$, with all operators allowed by the symmetries present in K_{eff} and W_{eff}. However, it turns out that supersymmetry decrees

$$W_{\text{eff}} = W\,. \tag{2.2}$$

This is called a *non-renormalization theorem*. In words,

The superpotential is not affected by renormalization. All renormalization effects can be absorbed into the effective Kähler potential.

This statement holds also in more general theories involving chiral superfields, at least in perturbation theory.

There are some immediate implications for supersymmetric theories:

- If a relevant or marginal coupling is not present in the defining UV theory, then it will not be generated radiatively, even if there is no symmetry which protects it.
- All divergences are due to wave-function renormalization.
- All divergences are therefore at most logarithmic. *There are no quadratic divergences in supersymmetry.*

Altogether supersymmetric theories exhibit a much better UV behaviour than generic QFTs.

The non-renormalization theorem can be proved in several ways. The original proof used a supersymmetric generalization of Feynman diagrams, which have somewhat fallen out of use since. Later, when discussing supersymmetric versions of the Standard Model, we will work out more explicitly how quadratic divergences cancel between fermionic and bosonic loops. As for the non-renormalization theorem, we will now give a much more elegant

Proof of non-renormalization using holomorphy

We are following Seiberg (1993) for an argument based solely on symmetry and holomorphy. We take a renormalizable model of just a single superfield as an example, but the proof can be easily generalized to general Wess-Zumino models (see e.g. Weinberg's book).

A theory of a single chiral superfield and vanishing superpotential, $W = 0$, is invariant under two obvious U(1) symmetries which separately rotate the phases of the spinor and the scalar. We can form two linear combinations out of these phase rotations which we call U(1) and $\text{U}(1)_R$. The U(1) symmetry simply sends

$$\Phi \to e^{-i\alpha}, \qquad \Phi^\dagger \to e^{i\alpha} \tag{2.3}$$

for some constant α. The $\mathrm{U}(1)_R$ symmetry is an R-symmetry which does not commute with supersymmetry (see Section 1.2.3):

$$[Q_\alpha, R] = Q_\alpha \tag{2.4}$$

(since we are in $\mathcal{N} = 1$ SUSY, any R-symmetry can be at most a phase rotation). Therefore $\mathrm{U}(1)_R$ acts differently on the components of Φ. Explicitly, with $\Phi = \phi + \sqrt{2}\,\theta\psi + \theta^2\,F + \ldots$ we find for the $\mathrm{U}(1)_R$ charges of ϕ, ψ and F

$$R[\phi] = R[\psi] + 1 = R[F] + 2\,. \tag{2.5}$$

It is often useful to think of the entire superfield Φ as carrying an R-charge, conventionally chosen to be that of its lowest component, $R[\Phi] \equiv R[\phi]$. To account for the transformations of its higher components, one assigns an R-charge also to the θ coordinates, $R[\theta] = 1$. With these conventions, for the superpotential Lagrangian

$$\int \mathrm{d}^2\theta\, W$$

to be R-symmetric, the superpotential needs to carry R-charge 2 (for a theory with vanishing superpotential this is of course immaterial). We can assign the $\mathrm{U}(1) \times \mathrm{U}(1)_R$ charges as follows:

	$\mathrm{U}(1)$	$\mathrm{U}(1)_R$
Φ	1	1

Next, we introduce a superpotential. The most generic renormalizable superpotential is

$$W_{\mathrm{tree}} = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3 \tag{2.6}$$

(a constant term in W does not affect the Lagrangian, and a linear term can generically be eliminated by a field redefinition). This superpotential breaks $\mathrm{U}(1) \times \mathrm{U}(1)_R$ explicitly. The first term is a mass term for Φ which breaks $\mathrm{U}(1)$, and the second term is an interaction term which also breaks $\mathrm{U}(1)_R$. The theory for which we would like to prove the non-renormalization theorem is the one defined by Eq. (2.6) and canonical $K = |\Phi|^2$ (it is straightforward, again, to generalize the argument to non-renormalizable interactions). That is, we seek to find the infrared effective superpotential W_{IR} at some scale μ and show that it equals W_{tree}.

We can write down an auxiliary theory with two more chiral superfields M and L:

$$W_{\mathrm{aux}} = \frac{M}{2}\Phi^2 + \frac{L}{3}\Phi^3\,. \tag{2.7}$$

The auxiliary theory is invariant under $\mathrm{U}(1) \times \mathrm{U}(1)_R$ with the charge assignments

	$\mathrm{U}(1)$	$\mathrm{U}(1)_R$
Φ	1	1
M	-2	0
L	-3	-1

Giving M and L vacuum expectation values, $\langle M\rangle = m$ and $\langle L\rangle = \lambda$, restores the original theory (2.6). The important point is that the low-energy effective theories are *indistinguishable*. It is impossible to tell, at low energies, if the theory was defined with the symmetries explicitly broken to start with, or as the low-energy theory after integrating out fields L and M whose expectation values break the symmetries spontaneously.

However, by supersymmetry, the low-energy effective superpotential for the theory Eq. (2.7) depends holomorphically on the fields Φ, L, and M. Therefore,

$$W_{\mathrm{aux,IR}} = M\Phi^2\, f\left(\frac{L\Phi}{M}\right) \tag{2.8}$$

where f is some holomorphic function to be determined. Any dimension-3, U(1)-invariant superpotential of R-charge 2 must take this form. It follows that

$$W_{\mathrm{IR}} = m\Phi^2\, f(x)\,, \qquad x \equiv \frac{\lambda\Phi}{m}\,. \tag{2.9}$$

In other words, the infrared superpotential for our original theory depends holomorphically not just on the fields but also on the couplings (which can be regarded as descending from the vacuum expectation values of L and M in the auxiliary theory). To determine the function f, we consider the weak-coupling limit in which λ is arbitrarily small. Notice that any x can be reached in this limit by an appropriate choice of m. In the weak coupling limit, W_{IR} must approach the tree-level superpotential, $W_{\mathrm{IR}} \rightarrow W_{\mathrm{tree}}$. Therefore,

$$f(x) = 1 + x \tag{2.10}$$

and we conclude

$$W_{\mathrm{IR}} = W_{\mathrm{tree}}\,. \tag{2.11}$$

The exact superpotential is fixed by holomorphy, asymptotics, and symmetries alone.

This will be a recurring theme during the next few sections.

Exercise 13: Global U(1)$_R$ symmetries and spontaneous SUSY breaking

Consider the O'Raifeartaigh model of three chiral superfields Φ_1, Φ_2, and X. The superpotential is

$$W = \lambda\Phi_1(X^2 - v^2) + m\Phi_2 X\,.$$

a. Show that this theory is invariant under a global U(1)$_R$ symmetry, and find the R-charges of Φ_1, Φ_2 and X.

b. Show that W is the most general renormalizable superpotential allowed by U(1)$_R \times Z_2$, where Z_2 is a suitable (non-R) parity.

c. Assuming that $2\lambda^2 v^2 < m^2$, find the classical vacua, show that supersymmetry is spontaneously broken, and show that U(1)$_R$ is also spontaneously broken almost everywhere.

d. Add a renormalizable term which explicitly breaks U(1)$_R$, and show that there is now a supersymmetric vacuum.

2.1.2 Supersymmetric QCD

We now turn to a more interesting theory, namely supersymmetric QCD. SQCD is defined, for our purposes, as a supersymmetric $\mathrm{SU}(N_c)$ gauge theory with N_f flavours of "quark" and "antiquark" superfields in the fundamental and antifundamental representation. (Real-world QCD has of course $N_c = 3$, $N_f = 6$, and is missing the fermionic "gluino" partners of the gauge bosons as well as the scalar "squark" partners of the quark fields.) The Lagrangian is

$$\mathcal{L} = \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,\left(Q_i^\dagger e^V Q_i + \widetilde{Q}_i^\dagger e^{-V}\widetilde{Q}_i\right) - \frac{1}{4\,g^2}\int \mathrm{d}^2\theta\, W^{a\alpha}W^a_\alpha + \text{h.c.}\,. \tag{2.12}$$

Notice that we have moved the gauge coupling into the normalization of the kinetic term. We are ignoring the vacuum angle for the moment.

The gauge coupling g is affected by loop corrections. This leads to it becoming scale dependent, as discussed above. Its scale dependence is described by the β function,

$$\frac{\mathrm{d}}{\mathrm{d}\log\mu} g(\mu) = \beta(g(\mu))\,. \tag{2.13}$$

As can be found in any QFT textbook, at one loop the β function is

$$\beta^{(1)}(g) = -\frac{b}{16\pi^2} g^3\,, \tag{2.14}$$

where the *one-loop beta function coefficient* b depends on the gauge group and matter content (for fields of spin j in representations r) as

$$b = -\frac{1}{2} c_{\mathrm{Ad},1} - \frac{1}{2} c_{\mathrm{Ad},0} + \frac{1}{2}\sum_{\text{matter}} c_{r,j} \tag{2.15}$$

and where

$$c_{r,j} = T(r)\times\begin{cases} -2/3; & j = 0,\ \text{scalars} \\ -4/3; & j = 1/2,\ \text{Weyl fermions} \\ -20/3; & j = 1,\ \text{vectors}\end{cases} \tag{2.16}$$

Here $T(r)$ is half the *Dynkin index* of the representation r, defined for the gauge generators T^a as

$$\mathrm{tr}_r\, T^a T^b = T(r)\,\delta^{ab}\,. \tag{2.17}$$

With $T(\mathbf{Ad}) = N_c$ for the adjoint of $\mathrm{SU}(N_c)$, and $T(\mathbf{N}_c) = T(\overline{\mathbf{N}}_c) = \frac{1}{2}$ for the fundamental and the antifundamental, we get

$$\begin{aligned}\frac{10}{3}N_c + \frac{1}{3}N_c + \frac{1}{2}\cdot\left(-\frac{4}{3}N_c - 2\cdot\left(-N_f\cdot\frac{4}{3}\cdot\frac{1}{2} - N_f\cdot\frac{1}{3}\cdot\frac{1}{2}\right)\right) \\ = \boxed{b = 3\,N_c - N_f}\,.\end{aligned} \tag{2.18}$$

The first term comes from the gauge bosons, the second from the ghost contributions, the first term in parentheses from the gaugino, and the remaining terms from quarks, antiquarks, squarks, and antisquarks. Since $b > 0$ for $3\,N_c > N_f$, SQCD can be asymptotically free if the number of flavours is not too large. In that case, the theory is well-defined in the UV, but similar to ordinary QCD it will become non-perturbative at large distances. For $3\,N_c < N_f$, asymptotic freedom is lost. In that case the theory makes sense as an effective theory in the far infrared, but it will become strongly coupled in the UV.[1]

We are now ready to establish another renormalization theorem, this time for the SQCD gauge coupling. To this end, we write the super-Yang-Mills part of the Lagrangian somewhat differently, this time including the ϑ angle:

$$\mathcal{L}_{\mathrm{SYM}} = \frac{1}{16\pi\, i} \int \mathrm{d}^2\theta \, S\, W^{a\alpha} W^a_\alpha + \text{h.c.} \tag{2.19}$$

In general theories, S will be a chiral superfield (which may depend holomorphically on other chiral superfields in the theory). In SYM theory, it is given by

$$S = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g^2} \tag{2.20}$$

where g is the running gauge coupling, and ϑ plays the same role as the CP-violating ϑ angle in ordinary QCD. In components,

$$\begin{aligned} \mathcal{L}_{\mathrm{SYM}} =& -\frac{1}{4\, g^2}\left(F^{a\mu\nu}F^a_{\mu\nu} + 4i\,\lambda^a\sigma^\mu\partial_\mu\bar{\lambda}^a + 2\, D^a D^a\right) - \frac{\vartheta}{64\pi^2}\epsilon_{\mu\nu\kappa\lambda}F^{a\mu\nu}F^{a\kappa\lambda} \\ \equiv& \, \mathcal{L}_{\mathrm{CP}} + \mathcal{L}_{\not{\mathrm{CP}}}\,. \end{aligned} \tag{2.21}$$

The CP-violating part is a total derivative: Denoting by tr the trace in the fundamental representation, one can write (→ Exercises)

$$\mathcal{L}_{\not{\mathrm{CP}}} = -\frac{\vartheta}{8\pi^2}\epsilon_{\mu\nu\kappa\lambda}\partial^\mu \operatorname{tr}\left(A^\nu\partial^\kappa A^\lambda + \frac{2}{3}A^\nu A^\kappa A^\lambda\right)\,. \tag{2.22}$$

However, since the gauge field A_μ need not vanish at infinity (it should only approach a pure gauge configuration, such that the field strength vanishes) there can be boundary contributions. Indeed, in instanton calculus (and differential topology) it is a standard result that

$$\frac{1}{64\pi^2}\int \mathrm{d}^4x\, \epsilon_{\mu\nu\kappa\lambda}F^{a\mu\nu}F^{a\kappa\lambda} \equiv n \in \mathbb{Z} \tag{2.23}$$

is an integer topological quantum number, called the *winding number*, *Pontryagin index* or *instanton number*. It follows that the path integral

$$\int [\mathcal{D}\phi]\, e^{-i\int \mathrm{d}^4x \mathcal{L}_{\mathrm{CP}} - in\vartheta}$$

is invariant under the shift $\vartheta \to \vartheta + 2\pi$: physics is periodic in ϑ.

Now let us return to the running gauge coupling. We can solve the one-loop renormalization group equation to obtain

$$\frac{\mathrm{d}g}{\mathrm{d}\log\mu} + -\frac{b}{16\pi^2}g^3 \;\Rightarrow\; \frac{1}{g^2(\mu)} = -\frac{b}{8\pi^2}\log\frac{|\Lambda|}{\mu}\,. \tag{2.24}$$

[1] Very remarkably, for certain numbers of flavours and colours these two configurations can be *dual* to each other (*Seiberg duality*). Fleshing out this duality requires pursuing the subject of SQCD just a little more than we will have time to in these lectures, but the reader is very much encouraged to study it e.g. using Terning's book, or Peskin's lecture notes. After our Section 2.1.4 we will have covered all necessary background.

Here $|\Lambda|$ is the strong-coupling scale of the theory, such that $g(\mu = |\Lambda|) = 1$. This allows us to define a one-loop effective gauge-kinetic function,

$$S_{1-\text{loop}} = \frac{b}{2\pi i} \log \left(\frac{|\Lambda|}{\mu} e^{i\vartheta} \right) , \tag{2.25}$$

which in terms of the *holomorphic scale*

$$\Lambda \equiv |\Lambda| e^{i\vartheta/b} \tag{2.26}$$

is equivalent to

$$S_{1-\text{loop}} = \frac{b}{2\pi i} \log \frac{\Lambda}{\mu} . \tag{2.27}$$

Our goal is now to find the higher-order loop corrections to S; we will eventually find that they vanish. To show this, observe that the transformation

$$\Lambda \to \Lambda\, e^{2\pi i/b}$$

amounts to

$$\vartheta \to \vartheta + 2\pi, \qquad S \to S + 1 \tag{2.28}$$

under which the physics does not change. Therefore

$$S = \frac{b}{2\pi i} \log \frac{\Lambda}{\mu} + (\text{terms invariant under } \Lambda \to \Lambda\, e^{2\pi i/b}) \tag{2.29}$$

as the one-loop term already accounts for the shift of Eq. (2.28). Since at weak coupling ($\Lambda \to 0$), we should recover the leading one-loop result, the invariant terms are regular and the effective gauge-kinetic function at the scale μ takes the form

$$S(\Lambda, \mu) = \frac{b}{2\pi i} \log \frac{\Lambda}{\mu} + \sum_{n=1}^{\infty} c_n \left(\frac{\Lambda}{\mu} \right)^{nb} . \tag{2.30}$$

The polynomial terms are not analytic in g; they correspond to non-perturbative n-instanton contributions with unknown coefficients c_n. As long as g remains perturbatively small, they are strongly suppressed.

We have established another non-renormalization theorem:

> The holomorphic gauge coupling is not renormalized in perturbation theory beyond one loop.

However, some care must be taken when interpreting this result. The actual physical gauge coupling is not equal to the holomorphic one, but is related by field rescalings $A^a_\mu \to g\, A^a_\mu$, $\lambda^a \to g\, \lambda^a$, $D^a \to g\, D^a$. This rescaling is in general anomalous. Furthermore, our result is valid for the Wilsonian holomorphic coupling. If there are infrared divergences, the connection with the physical gauge coupling is subtle, because the infrared divergences need to be regularized and the dependence on the regulators will be non-holomorphic. As a consequence of all this, the physical gauge coupling does, in general, receive contributions from higher loop orders. They can be subsumed in a compact all-orders formula for the SQCD β function called the Novikov-Shifman-Vainshtein-Zakharov (NSVZ) β function (which still depends on the anomalous dimensions of the matter fields).

In $\mathcal{N} = 2$ supersymmetric gauge theory, the c_n coefficients in Eq. (2.30) can be exactly computed, such that the entire gauge-kinetic function is known. Since in $\mathcal{N} = 2$ this data is enough to fix also the Kähler potential, this means that $\mathcal{N} = 2$ SQCD is exactly solvable in the infrared. This is the celebrated Seiberg-Witten theory, which unfortunately we will have no time to discuss in detail.

Exercise 14: The topological term in the Yang-Mills action

Verify Eq. (2.22).

2.1.3 Gaugino condensation in super-Yang-Mills theory

Let us consider pure supersymmetric Yang-Mills theory, setting $N_f = 0$. This is still an interesting theory since the gauge fields interact with themselves and with the gaugino. At the classical level, there is a global $\mathrm{U}(1)_R$ R-symmetry under which the field strength W^a_α has charge 1. More precisely, it acts by a phase rotation on the lowest component of W^a_α (which is the gaugino λ^a_α, see Eq. (1.184)) while not affecting the θ-component (which is essentially the field strength $F^a_{\mu\nu}$).

It is a standard result that the chiral rotation

$$\lambda^a_\alpha \to e^{i\beta}\lambda^a_\alpha\,, \qquad\qquad \beta \text{ constant}, \tag{2.31}$$

is *anomalous* in a theory of a single chiral fermion λ. This means that, while $\mathrm{U}(1)_R$ is a perfectly good symmetry of the classical theory, there exists no regulator which preserves it: The R-symmetry will be broken by quantum corrections. A simple way to make this more quantitative is to consider the sourceless partition function for the fermion in a classical gauge background, regarded as a functional of the background gauge field:

$$\mathcal{Z}[A^a_\mu] = \int [\mathcal{D}\lambda][\mathcal{D}\bar\lambda]\, e^{i\mathcal{S}[A^a_\mu,\lambda^a_\alpha,\bar\lambda^a_{\dot\alpha}]} \tag{2.32}$$

where the classical action is

$$\mathcal{S}[A^a_\mu,\lambda^a_\alpha,\bar\lambda^a_{\dot\alpha}] = -\frac{1}{g^2}\int \mathrm{d}^4x\, i\lambda\sigma^\mu D_\mu\bar\lambda\,. \tag{2.33}$$

The classical action is evidently invariant under the chiral rotation of Eq. (2.31), but it turns out that the *path integral measure* is not. The measure transforms with a Jacobian which needs to be regularized, and it turns out that this Jacobian is non-trivial in a gauge field background:

$$\int [\mathcal{D}\lambda][\mathcal{D}\bar\lambda] \to \int [\mathcal{D}\lambda][\mathcal{D}\bar\lambda] e^{i\int \mathrm{d}^4x\,\mathcal{A}(A^a_\mu)}\,. \tag{2.34}$$

Here the *(covariant) anomaly* $\mathcal{A}$ is

$$\mathcal{A}(A^a_\mu) = \frac{\beta}{32\pi^2}\,\epsilon_{\mu\nu\kappa\lambda}\,\mathrm{tr}\,F^{\mu\nu}F^{\kappa\lambda}\,, \tag{2.35}$$

and the trace should be taken in the representation in which the fermion transforms; for the gaugino in super-Yang-Mills theory, this is the adjoint, which implies

$$\mathcal{A}(A^a_\mu) = \frac{\beta}{32\pi^2}\,T(\mathbf{Ad})\,\epsilon_{\mu\nu\kappa\lambda}F^{b\mu\nu}F^{b\kappa\lambda} = \frac{\beta N_c}{32\pi^2}\,\epsilon_{\mu\nu\kappa\lambda}F^{b\mu\nu}F^{b\kappa\lambda}\,. \tag{2.36}$$

Under Eq. (2.31) the generating functional $\mathcal{Z}$ then transforms as

$$\int [\mathcal{D}\lambda][\mathcal{D}\bar{\lambda}]\, e^{i\mathcal{S}[A^a_\mu,\lambda^a_\alpha,\bar{\lambda}^a_{\dot{\alpha}}]} \rightarrow \int [\mathcal{D}\lambda][\mathcal{D}\bar{\lambda}]\, e^{i\mathcal{S}[A^a_\mu,\lambda^a_\alpha,\bar{\lambda}^a_{\dot{\alpha}}]+i\int \mathrm{d}^4x\,\mathcal{A}} . \tag{2.37}$$

Comparing with Eq. (2.21), we see that a chiral rotation corresponds to a shift in ϑ:

$$\vartheta \rightarrow \vartheta - 2\beta N_c . \tag{2.38}$$

This is only a symmetry if $\beta N_c \pi$ is integer, or equivalently if

$$\beta = \frac{k\pi}{N_c} \qquad\qquad k \in \mathbb{Z} . \tag{2.39}$$

Therefore, in the quantum theory the classical $\mathrm{U}(1)_R$ symmetry is broken to a Z_{2N_c} *discrete subgroup.*

Recall that in Section 2.1.1 we promoted the parameters of our theory to chiral superfields, in order to prove a non-renormalization theorem. These parameters were assigned charges under the symmetries of the free theory, with the symmetries being broken by nonzero expectation values of these chiral superfields. We can do something similar here. Let us promote the strong-coupling scale Λ to a chiral superfield, and give it an R-charge $R[\Lambda] = \frac{2}{3}$. Then, under a $\mathrm{U}(1)_R$ transformation,

$$\lambda \rightarrow e^{i\beta}\lambda \qquad\qquad \Lambda \rightarrow e^{\frac{2}{3}i\beta}\Lambda = |\Lambda| e^{i(\frac{\vartheta}{3N_c}+\frac{2}{3}\beta)} \tag{2.40}$$

(compare the definition of Λ in Eq. (2.26), and note that $b = 3N_c$ in pure super-Yang-Mills). The generating functional is now superficially invariant under $\mathrm{U}(1)_R$ – of course the R-symmetry is still broken, but the breaking can now be regarded as a spontaneous breaking due to the non-vanishing Λ.

We now want to know what happens to the theory in the far infrared. In analogy to ordinary QCD we expect super-Yang-Mills theory to have a *mass gap*, i.e. no dynamical excitations below a certain positive energy threshold. In other words, the massless gauge fields and gauginos should form massive bound states at energies below the strong-coupling scale. If this is the case, then the action in the deep infrared is trivial, and the low-energy superpotential can at most be a constant. From the anomalous R-symmetry, since W should have R-charge 2, it follows that the only possible term is[2]

$$W_{\mathrm{IR}} = a\,\Lambda^3 . \tag{2.41}$$

Here a is a renormalization-scheme dependent constant (which can in principle be calculated using instantons; for instance, $a = N_c$ in the $\overline{\mathrm{DR}}$ renormalization scheme). Of course this superpotential is not very interesting as such. However, it can play an important role in more complicated theories, which involve super-Yang-Mills theory coupled to other dynamical fields. For instance, a constant term in the superpotential can be very important in supergravity; and in theories where the gauge coupling is given by the expectation value of a dynamical field S, Eq. (2.41) can be interpreted as a proper superpotential term for S.

We can extract some more information from these considerations. In the vacuum, the expectation value of the gaugino bilinear $\langle\lambda^a\lambda^a\rangle$ can be non-vanishing. Treating the

[2]We could as well have guessed this from dimensional analysis alone, as Λ is the only mass scale in the game and the superpotential should have mass dimension 3.

gauge-kinetic function S as a background chiral superfield, its F-component F_S will act as a source for $\lambda^a\lambda^a$ since

$$\mathcal{L} = \frac{1}{16\pi i}\int \mathrm{d}^2\theta S\, W^{a\alpha}W^a_\alpha + \text{h.c.} = \frac{1}{16\pi i}\,F_S\,\lambda^a\lambda^a + \text{h.c.} + \ldots \tag{2.42}$$

We can obtain the expectation value $\langle\lambda^a\lambda^a\rangle$ as usual by differentiating the generating functional with respect to the source:

$$\langle\lambda^a\lambda^a\rangle = 16\pi i\frac{\partial}{\partial F_S}\log\mathcal{Z} = 16\pi i\,\frac{\partial}{\partial F_S}\mathcal{S} = 16\pi i\,\frac{\partial}{\partial F_S}\int \mathrm{d}^2\theta\, W_{\mathrm{IR}} = 16\pi i\,\frac{\partial}{\partial S}W_{\mathrm{IR}}\bigg|_{\theta=0}. \tag{2.43}$$

Using Eq. (2.30), with $b = 3N_c$, and dropping the non-perturbative contributions (which only contribute a phase), we have

$$W_{\mathrm{IR}} = a\Lambda^3 = a\mu^3 e^{\frac{2\pi i}{N_c}S} \tag{2.44}$$

and thus

$$\boxed{\langle\lambda^a\lambda^a\rangle = -\frac{32\pi^2}{N_c}a\,\Lambda^3\,.} \tag{2.45}$$

Thus the vacuum of super-Yang-Mills theory is indeed permeated by a non-vanishing gaugino condensate.

We have seen above that the anomaly-free subgroup of $\mathrm{U}(1)_R$ is the discrete subgroup Z_{2N_c}. By Eq. (2.45) this symmetry is spontaneously broken in the vacuum, since the gaugino condensate is invariant only under the Z_2 subgroup generated by $\lambda^a \to -\lambda^a$.

Exercise 15: The holomorphic gauge coupling revisited

Repeat the proof of the non-renormalization theorem for the holomorphic gauge coupling in super-Yang-Mills theory, using the anomalous $\mathrm{U}(1)_R$ symmetry. To do so, promote the holomorphic scale Λ to a chiral superfield of R-charge $2/3$. Then show that demanding

- the gauge-kinetic function S should be holomorphic in Λ
- and the theory should be invariant under the spurious $\mathrm{U}(1)_R$ symmetry, at the perturbative level,

fixes the perturbative part of S to be given by the one-loop expression of Eq. (2.27).

Hint: Write down how the (perturbative part of the) holomorphic function $S(\Lambda)$ should transform under an R-rotation, and differentiate what you get with respect to the transformation parameter.

2.1.4 The Affleck-Dine-Seiberg superpotential

(To be completed)

Exercise 16: The ADS superpotential

In this exercise we repeat the steps of the derivation of the ADS super potential. This is the low energy effective action of a theory with N_f paris of massless chiral superfields Q and $\widetilde{Q}$ and their conjugates, coupled to $\mathrm{SU}(N_c)$ Yang-Mills gauge theory.

a. By writing down the outline of the full action, argue this theory enjoys a $\mathrm{U}(N_f) \times \mathrm{U}(N_f)$ flavor symmetry.

b. Argue that the $\mathrm{U}(1) \times \mathrm{U}(1)$ subgroup of this group can be split into a chiral $\mathrm{U}(1)_A$ and a non-chiral $\mathrm{U}(1)_B$. Which fermions are charged under the chiral symmetry?

c. Argue that in addition this theory enjoys a $\mathrm{U}(1)_R$ symmetry by assigning an R-charge to the superfields. Which fermions are charged under the R-symmetry?

d. The chiral symmetries are anomalous. Argue that they can be made non-anomalous by assigning a transformation rule to the ϑ angle. Show that the charge of the ϑ angle depends linearly on the number of fermions charged under each chiral U(1).

e. Argue that $\det T_{i\bar{i}} \equiv \det(\langle Q_{i,a}\widetilde{Q}^a_{\bar{i}}\rangle)$ for $N_f < N_c$ is the natural gauge and Lorentz-invariant order parameter of the theory at low energies, in addition to $\Lambda^{3N_c-N_f}$.

f. Find the additional order parameters if $N_f = N_c$.

g. Verify the following table where $Q, \widetilde{Q}$ are the fermionic fields out of the chiral multiplets and λ is the gaugino:

field	$\mathrm{U}(1)_A$	$\mathrm{U}(1)_R$
Q	1	1
$\widetilde{Q}$	1	-1
λ	0	1
$\Lambda^{3N_c-N_f}$	$2N_f$	$2N_f-2N_c$
$\det T$	$2N_f$	0

h. Find the ADS superpotential as the unique combination of Λ and $\det T$ which has R-charge 2 and $\mathrm{U}(1)_A$ charge 0.

Exercise 17: More on the ADS superpotential

The ADS superpotential for the composite field T does not lead to a stable vacuum by itself. However, we can construct more complicated models which do have a vacuum state and where it plays an important role. This exercise gives a simple example. Consider a theory of chiral superfields Q^i_a and $\widetilde{Q}^a_i$, where $a = 1 \dots N_c$ and $i = 1 \dots N_f$, with $N_f < N_c$. We also introduce a chiral superfield S, and take the tree-level superpotential to be

$$W_{\text{tree}} = \lambda\, S Q^i_a \widetilde{Q}^a_i + \kappa\, S^3 \,.$$

a. Show that this theory, with no gauge fields involved, has an $\mathrm{SU}(N_f) \times \mathrm{SU}(N_c) \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R$ symmetry. Find its vacuum states, and identify the unbroken symmetries.

b. Gauge the $\mathrm{SU}(N_c)$ symmetry, and write down the low-energy superpotential for S and for the composite fields $T^i_j \equiv Q^i_a \widetilde{Q}^a_j/\Lambda$ (where Λ is the strong-coupling scale of $\mathrm{SU}(N_c)$). Find the vacuum states of this theory, and identify the unbroken symmetries.

2.2 Supersymmetry and TeV-scale particle physics

One of the main reasons for the popularity of supersymmetry is the hope to find evidence for supersymmetry in real-world physics, in particular, at the LHC collider. If particle physics is supersymmetric, SUSY must be spontaneously broken (otherwise we would observe all particles coming in complete supermultiplets, with mass-degenerate fermionic and bosonic components, which is clearly not the case). The scale of supersymmetry breaking is related to the mass splittings between bosons and fermions; if the superpartner masses are of the order of the electroweak scale or somewhat larger (but not too far above a TeV) then superpartners can be produced at the LHC. The key argument for superpartners to have at most TeV-scale masses is that, if supersymmetry breaking is connected to electroweak symmetry breaking, it can solve the *electroweak hierarchy problem.* In the following lectures we will introduce the hierarchy problem and use it to motivate a supersymmetric extension of the Standard Model.

2.2.1 The electroweak hierarchy problem

Let us recall the essential features of the Standard Model of particle physics: The SM is a gauge theory with gauge group SU(3) $\times$ SU(2) $\times$ U(1). Charged under this gauge group are three generations of chiral fermions, *quarks* which charged under SU(3) and *leptons* which are neutral under SU(3). There is also a scalar *Higgs field* whose vacuum expectation value spontaneously breaks SU(2) $\times$ U(1) to its diagonal U(1) subgroup. The scale of this *electroweak symmetry breaking* is $v = 246$ GeV, and is set by the mass and coupling parameters in the Higgs scalar potential.

While the Standard Model describes almost all particle physics processes accurately, as far as they are accessible in the laboratory, there is a number of arguments which suggest that it should be embedded into a more fundamental theory. Some of these are motivated from cosmology, and others by purely theoretical considerations. Let us list just a few, in random order:

- The Standard Model does not account for dark matter, which according to standard cosmology constitutes about 80% of all matter in the universe.
- Neither does it accound for dark energy, which according to standard cosmology is responsible for most of the universe's energy density.
- The Standard Model cannot explain the observed baryon asymmetry, i.e. the prevalence of matter over antimatter in the universe.
- While neutrino masses can easily be accommodated in the Standard Model, explaining their smallness points to physics beyond the Standard Model.
- The SU(3) $\times$ SU(2) $\times$ U(1) gauge group with the peculiar charge assignments to matter fields looks somewhat arbitrary. In some extensions of the Standard Model, the forces and charges are conjectured to originate naturally from a single simple gauge group (*Grand Unification*).
- A consistent embedding of the Standard Model into a quantum theory of gravity

requires physics (far) beyond the Standard Model — indeed, very likely beyond the realm of weakly coupled QFT itself.

By all these arguments, some New Physics should appear above some energy scale $\Lambda_{\rm NP}$, with the SM being merely the low-energy effective field theory of a more fundamental theory. If this more fundamental theory is a weakly coupled QFT, we can think of $\Lambda_{\rm NP}$ as the mass scale of the New Physics states we have integrated out. Many New Physics models point to scales far above the electroweak scale, $\Lambda_{\rm NP} \gg v$. For instance,

- $\Lambda_{\rm NP} = M_{\rm GUT} \approx 10^{16}$ GeV is the scale associated with Grand Unification,
- $\Lambda_{\rm NP} = M_{\rm Planck} \approx 10^{19}$ GeV is the scale associated with quantum gravity.

The *electroweak hierarchy problem* is essentially the question:

> Why is the electroweak symmetry breaking scale so small compared to the fundamental scale?

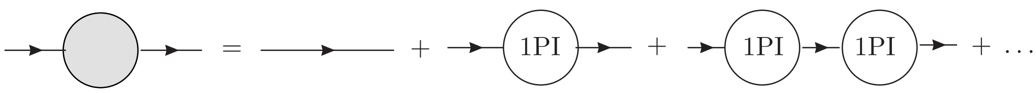
At first this may seem to be a merely philosophical question — the value of the electroweak scale being given by some parameters in the theory, any choice of parameters might seem as good as any other. To appreciate why the problem is indeed more serious than that, we need to look into the issue of *quadratic divergences* in QFT. We will do so following the review by Drees [11], and start by comparing two systems with very different renormalization properties.

Radiative corrections I: Electron mass in QED

Recall that the *electron self-energy* in QED, at some fixed external momentum p, is given by the sum of one-particle irreducible insertions into the bare propagator:

$$-\, i\Sigma(\not{p}) = \sum (\text{1PI insertions})\ . \tag{2.46}$$

The full propagator is then given by a geometric series:

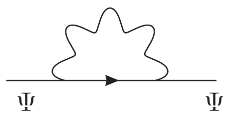


$$\begin{aligned} &= \frac{i}{\not{p}-m_0} + \frac{i}{\not{p}-m_0}\frac{\Sigma(\not{p})}{\not{p}-m_0} + \frac{i}{\not{p}-m_0}\left(\frac{\Sigma(\not{p})}{\not{p}-m_0}\right)^2 + \ldots \\ &= \frac{i}{\not{p}-m_0}\sum_n \left(\frac{\Sigma(\not{p})}{\not{p}-m_0}\right)^n \\ &= \frac{i}{\not{p}-m_0-\Sigma(\not{p})}\ . \end{aligned} \tag{2.47}$$

At zero external momentum, this leads to *mass renormalization*, where the tree-level mass m_0 is shifted as

$$m_0 \;\to\; m_0 + \Sigma(0)\,. \tag{2.48}$$

For the electron in QED, at one loop, the only 1PI contribution to the 2-point function is

$\Sigma^{(1-\mathrm{loop})}(0) =$ 

$$
\begin{aligned}
&= \int \frac{\mathrm{d}^4 q}{(2\pi)^4} (-ie\gamma_\mu) \frac{i}{\not{p} - m_0} (-ie\gamma_\nu) \frac{-i\eta^{\mu\nu}}{q^2} \\
&= -e^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\gamma_\mu (\not{p} + m_0) \gamma^\mu}{q^2 (q^2 - m_0^2)} \\
&= -4e^2 m_0 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{q^2 (q^2 - m_0^2)} \\
&= -\frac{e^2 m_0}{4\pi^2} \log \frac{\Lambda_{\mathrm{NP}}}{m_0} + \text{finite}\,.
\end{aligned}
\tag{2.49}
$$

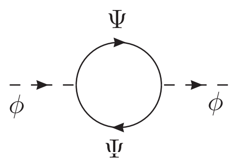
Here Λ is the UV cutoff, and the logarithmic divergence indicates some mild dependence of the electron mass on UV physics. However, even for a cutoff as large as the Planck scale, the quantum correction is very modest. Note in particular that it is proportional to the electron mass itself. This is an indication that a small electron mass, as compared to the cutoff scale, is "technically natural". Indeed, in the limit of vanishing m_0, there is an additional symmetry of the theory (chiral symmetry, under which $\Psi \to e^{i\alpha\gamma^5}\Psi$) which protects the electron mass and prevents it from being generated in perturbation theory. Any loop-induced electron mass must therefore be proportional to the (small) parameter which explicitly breaks the chiral symmetry, i.e. the bare electron mass itself. The coefficient is enhanced by a large logarithm $\log \Lambda_{\mathrm{NP}}/m_0$, but at the same time suppressed by a small loop factor.

Radiative corrections II: Higgs mass

Let us contrast this situtation with that of a scalar field such as the Higgs. Adding the Standard Model Higgs field to the above system, we can compute the mass renormalization it receives e.g. from electron (or general fermion) loops. Parameterizing the Higgs field as $h(x) = \frac{1}{\sqrt{2}}(v + \phi(x))$, with $\langle\phi\rangle = 0$, the electron and ϕ are coupled by a Yukawa interaction:

$$
\mathcal{L} = \frac{\lambda}{\sqrt{2}} \phi \overline{\Psi} \Psi + \ldots \tag{2.50}
$$

The fermion contribution to the one-loop scalar mass renormalization is

$M^{2\,(1-\mathrm{loop})}(0) =$ 

$$
\begin{aligned}
&= -\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\,\mathrm{tr}\left(i\frac{\lambda}{\sqrt{2}}\frac{i}{\not{p}-m_\Psi}i\frac{\lambda}{\sqrt{2}}\frac{i}{\not{p}-m_\Psi}\right)\\
&= -2\lambda^2\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{q^2+m_\Psi^2}{(q^2-m_\Psi^2)^2}\\
&= -2\lambda^2\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\left(\frac{1}{q^2-m_\Psi^2}+\frac{2\,m_\Psi^2}{(q^2-m_\Psi^2)^2}\right).
\end{aligned}
\tag{2.51}
$$

Here the overall minus sign and the Dirac trace appear because we are computing a fermion loop. The first term in the last row is *quadratically divergent*; it will give rise to a Higgs mass "correction"

$$
\delta m_\phi^2 \sim \mathcal{O}(1)\cdot\Lambda_{\mathrm{NP}}^2 \tag{2.52}
$$

which is huge if $\Lambda_{\mathrm{NP}} \gg m_\phi^2$. The quadratic divergence is an indication that various contributions from new physics are generically of the order of Λ_{NP} themselves, and would have to cancel to incredible precision in order to generate a Higgs mass in the low-energy effective theory which is many orders of magnitude below Λ_{NP}. If the cutoff scale is the Planck scale, then $m_\phi = 125$ GeV requires cancellations at the level of roughly one part in 10^{34}. Put differently, the Higgs mass is extremely sensitive to physics at much shorter distance scales. Since there is nothing to protect it, it would prefer to be close to the highest scale in the theory, which is the UV cutoff. This illustrates that a large hierarchy between the UV scale and the electroweak scale (which is tied to the Higgs mass) can only result from very unnatural, fine-tuned parameter choices. Here lies the essence of the hierarchy problem.

Solving the hierarchy problem

Evidently, the hierarchy problem would be solved if there was new physics not just at 10^{19} TeV but close to the electroweak scale, $\Lambda_{\mathrm{NP}} \sim 1$ TeV say. But then there would still be a hierarchy between our newly constructed New Physics extension of the Standard Model, whose characteristic scale is 1 TeV, and the grand-unified or see-saw neutrino or Planck scale. We are immediately faced with the question: What sets the scale of New Physics? There are essentially two ways out:

1. The scale of New Physics could be generated by dimensional transmutation, similar to QCD. In fact, there is no "QCD hierarchy problem", because the scale Λ_{QCD} arises through logarithmic running of the strong gauge coupling, and is therefore naturally exponentially below any fundamental UV scale such as the GUT or Planck scale. A similar thing could happen in whatever theory supersedes the Standard Model at energies around a TeV. There are several more or less constrained, more or less popular, more or less calculable examples for such models, which typically assert that the Higgs boson emerge as some composite state of a new gauge interaction becoming strongly coupled at a few TeV. There are even some which assert that there is in fact no Higgs boson, and that the resonance we observe is merely a relic of conformal symmetry breaking in some strongly coupled extra sector. We will not discuss this option any further in these lectures.
2. Whatever New Physics there is could be governed by supersymmetry. We have already mentioned that, in supersymmetric models, there are no quadratic diver-

gences, so there would not be a new hierarchy problem. Since these lectures are concerned with SUSY, we will go for this solution.

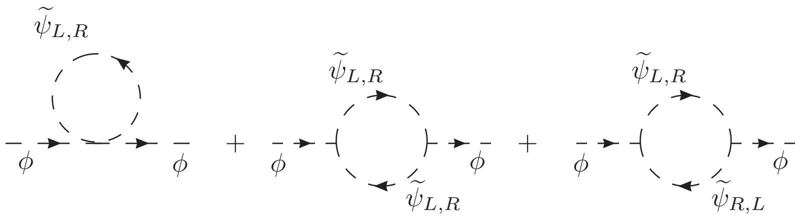
Radiative corrections III: Supersymmetry

It is instructive to see how exactly supersymmetry solves the problem of quadratic divergences. Let us take again the Higgs mass renormalization as an example, and add to our theory a pair of extra scalars $\widetilde{\psi}_L$ and $\widetilde{\psi}_R$. In a supersymmetric theory these would be the partners of the left-handed and of the right-handed electron. The Higgs is coupled to the electron and to the additional scalars through the terms

$$\mathcal{L} \supset \frac{\lambda}{\sqrt{2}}\phi\overline{\Psi}\Psi + \frac{\kappa}{2}\phi^2\left(|\widetilde{\psi}_L|^2 + |\widetilde{\psi}_R|^2\right) + \kappa\, v\,\left(|\widetilde{\psi}_L|^2 + |\widetilde{\psi}_R|^2\right)\phi + \left(\frac{\lambda}{\sqrt{2}}A\,\phi\widetilde{\psi}_L\widetilde{\psi}_R^* + \text{h.c.}\right) \tag{2.53}$$

Note that the relative factor between the $\phi^2|\widetilde{\psi}|^2$ and the $\phi|\widetilde{\psi}|^2$ terms has been fixed to be $2v$; this is because these terms originate from $|h|^2|\widetilde{\psi}|^2$ terms after electroweak symmetry breaking, where h is expanded as $h \sim (v+\phi)/\sqrt{2}$. In general the κ couplings could be different for $\widetilde{\psi}_L$ and $\widetilde{\psi}_R$, but with some hindsight we have chosen them equal. Also, pulling out a factor $\lambda/\sqrt{2}$ from the trilinear $\phi\widetilde{\psi}_L\widetilde{\psi}_R^*$ coupling is purely conventional.

The extra scalars will give the following contribution to the Higgs mass at one loop:



$$\begin{aligned}
&= -\kappa\int\frac{\mathrm{d}^4q}{(2\pi)^4}\left(\frac{1}{q^2-m^2_{\widetilde{\psi}_L}} + \frac{1}{q^2-m^2_{\widetilde{\psi}_R}}\right)\\
&\quad + (\kappa v)^2\int\frac{\mathrm{d}^4q}{(2\pi)^4}\left(\frac{1}{\left(q^2-m^2_{\widetilde{\psi}_L}\right)^2} + \frac{1}{\left(q^2-m^2_{\widetilde{\psi}_R}\right)^2}\right)\\
&\quad + \lambda^2A^2\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(q^2-m^2_{\widetilde{\psi}_L})(q^2-m^2_{\widetilde{\psi}_R})}
\end{aligned} \tag{2.54}$$

The terms in the first line are quadratically divergent. By comparing with Eq. (2.51), we see however that the quadratic divergence cancels agains the quadratically divergent electron contribution to the Higgs mass if we choose

$$\kappa = -\lambda^2\,. \tag{2.55}$$

Imposing the relation Eq. (2.55), it is instructive to also calculate the logarithmically divergent pieces from Eq. (2.54) and Eq. (2.51) explicitly. In the $\overline{\text{MS}}$ renormalization scheme we have

$$\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{q^2-m^2} = \frac{i}{16\pi^2}m^2\left(1-\log\frac{m^2}{\mu^2}\right) \tag{2.56}$$

and

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{\left(q^2 - m^2\right)^2} = -\frac{i}{16\pi^2} \log \frac{m^2}{\mu^2} \,. \tag{2.57}$$

Here μ is the renormalization scale. Assuming a common mass $m^2_{\widetilde{\psi}_L} = m^2_{\widetilde{\psi}_R} \equiv m^2_{\widetilde{\psi}}$ (for simplicity — this is not essential to the argument) and summing the contributions from Eqns. (2.51) and (2.54), we have

$$\begin{aligned} M^{2\,(1-\text{loop})}(p=0) = i\frac{\lambda^2}{16\pi^2} \Bigg(& - 2\, m_\Psi^2 \left(1 - \log \frac{m_\Psi^2}{\mu^2}\right) + 4\, m_\Psi^2 \log \frac{m_\Psi^2}{\mu^2} \\ & + 2\, m^2_{\widetilde{\psi}} \left(1 - \log \frac{m_{\widetilde{\psi}^2}}{\mu^2}\right) - 4\, m_\Psi^2 \log \frac{m^2_{\widetilde{\psi}}}{\mu^2} \\ & - A^2 \log \frac{m^2_{\widetilde{\psi}}}{\mu^2} \Bigg) \,. \end{aligned} \tag{2.58}$$

In the second line we have set $m_\Psi = \frac{\lambda}{\sqrt{2}} v$, which holds if the electron acquires its mass through electroweak symmetry breaking. The one-loop correction to the Higgs mass vanishes altogether if

$$m_{\widetilde{\psi}} = m_\Psi \,, \qquad A = 0 \,. \tag{2.59}$$

Let us summarize what we have found. In a theory where the Higgs field is coupled to a Dirac spinor and two complex scalars, the quadratically divergent pieces to the Higgs mass renormalization cancel if the couplings satisfy Eq. (2.55). There is no mass renormalization at all if, in addition, the spinor and scalar masses are equal and the scalar trilinear coupling A in Eq. (2.53) vanishes.

So far we haven't even mentioned supersymmetry, but the implication is obvious when comparing with what we know about supersymmetric component Lagrangians. In a supersymmetric theory, a Dirac spinor is necessarily accompanied by two complex scalars to form two chiral supermultiplets. Fermion and boson masses are necessarily degenerate, and the A-type coupling is forbidden because it cannot descend from a holomorphic superpotential. Eq. (2.55) holds, because both the Yukawa coupling and the scalar quartic coupling originate from the same cubic term in the superpotential. Therefore, our explicit calculation serves to confirm, for our example model, what we already knew from the non-renormalization theorem:

There is no mass renormalization in a supersymmetric theory.

In particular, there are no quadratic divergences. But we have shown even more: All divergences are at most logarithmic, even if supersymmetry is *explicitly broken* by a violation of Eqns. (2.59). What is needed for the absence of quadratic divergences is the right particle content, and that the marginal couplings are related by Eq. (2.55). This is true more generally, both beyond one-loop and for theories with a more general particle content. Such explicitly SUSY-violating terms which still preserve the benign UV properties of SUSY are called *soft SUSY-breaking terms*.

The theory remains free of quadratic divergences if supersymmetry is broken only by *soft terms*, which include non-supersymmetric scalar mass terms and trilinear A-terms.

Soft terms in general supersymmetric models were classified in the early 1980s, where it was found that, besides scalar masses and A-terms, Majorana gaugino masses are also soft. The fact that scalars and gauginos can be given soft masses has a quite remarkable implication: One can easily embed the Standard Model into a model with softly broken supersymmetry without getting into conflict with observation. In fact, there is a perfectly viable explanation why we haven't seen any superpartners of the Standard Model fermions and gauge bosons yet, which is that these are precisely the states that can be given large soft masses. By comparison, the masses of the Standard Model fermions and gauge bosons are due to electroweak symmetry breaking, and so are tied to the electroweak scale.

Furthermore, when such a supersymmetrized Standard Model is coupled to an external "hidden sector" which breaks supersymmetry not explicitly but spontaneously, this generically induces effective soft terms for the Standard Model superpartners:

Soft terms for Standard Model superpartners parameterize the effects of spontaneous supersymmetry breaking in some hidden sector.

Given an explicit model of the SUSY-breaking hidden sector, and of the states which couple it to the supersymmetrized Standard Model, one can in principle calculate the induced soft terms. However, to be as model-independent as possible, we will keep them as free parameters as we now introduce the Minimal Supersymmetric Standard Model.

2.2.2 The Minimal Supersymmetric Standard Model

To construct the Minimal Supersymmetric extension of the Standard Model (MSSM), we embed all Standard Model fieds into supermultiplets. All Standard Model gauge bosons need to be embedded into vector multiplets, and all chiral fermions into chiral supermultiplets. This gives the following particle content:

Gauge fields and gauginos:

spin-	1/2	1	SU(3)	SU(2)	U(1)
	λ_1	B_μ	**1**	**1**	0
	λ_2	W_μ	**1**	**3**	0
	λ_3	G_μ	**8**	**1**	0

Quarks, leptons, squarks and sleptons:

spin- 0	1/2	multiplet	SU(3)	SU(2)	U(1)
$\tilde{q}_I$	q_I	Q_I	$\mathbf{3}$	$\mathbf{2}$	1/6
$\tilde{u}_I$	u_I	U_I	$\overline{\mathbf{3}}$	$\mathbf{1}$	−2/3
$\tilde{d}_I$	d_I	D_I	$\overline{\mathbf{3}}$	$\mathbf{1}$	1/3
$\tilde{\ell}_I$	ℓ_I	L_I	$\mathbf{1}$	$\mathbf{2}$	−1/2
$\tilde{e}_I$	e_I	E_I	$\mathbf{1}$	$\mathbf{1}$	1

The I index is a generation index, running from 1 to 3. Note that we have written all chiral fermions in terms of left-handed Weyl spinors, such that we can easily embed them into left-chiral superfields; this means that u_I and d_I are actually the antiparticles of the usual right-handed quarks. One could also add right-handed neutrinos if desired, by including another three chiral superfields with quantum numbers $(\mathbf{1},\mathbf{1})_0$. We denote the *squark* and *slepton* scalar superpartners of the Standard Model quarks and leptons by a tilde ˜.

The lepton doublets and the conjugate Higgs doublet have the same quantum numbers, so it is tempting to identify the Higgs field as the superpartner of some Standard Model left-handed lepton, but it turns out that no realistic model of this kind can be constructed. As a first attempt we therefore introduce a separate chiral supermultiplet $H = (h, \tilde{h})$ with gauge quantum numbers $(\mathbf{1},\mathbf{2})_{1/2}$ to house the Higgs scalar. However, we are immediately faced with two problems.

1. The Higgs multiplet contains a Weyl spinor, the *higgsino*, which is charged under $\mathrm{SU}(2)\times\mathrm{U}(1)$. This induces $\mathrm{SU}(2)^2-\mathrm{U}(1)$ and $\mathrm{U}(1)^3$ gauge anomalies (the charges of the chiral fermions in each Standard Model generation are carefully balanced such that the anomalies cancel; adding additional chiral fermions will generically spoil this cancellation).

2. Some of the Standard Model Yukawa couplings are forbidden by supersymmetry. More precisely, the superpotential

$$W = y^{(u)}_{IJ}\, HQ_IU_J \tag{2.60}$$

(where $y^{(u)}$ is a 3×3 matrix) gives rise to the Yukawa couplings

$$\mathcal{L}_{\text{Yuk}} = y^{(u)}_{IJ}\, hq_Iu_J + \text{h.c.} \tag{2.61}$$

However, the Yukawa couplings for down-type quarks and leptons,

$$\mathcal{L}_{\text{Yuk}} = y^{(d)}_{IJ}\, h^\dagger q_Id_J + \text{h.c.} + y^{(e)}_{IJ}\, h^\dagger \ell_I e_J + \text{h.c.} \tag{2.62}$$

(where $y^{(d)}$ and $y^{(e)}$ are also 3×3 matrices) cannot be derived from a holomorphic superpotential.[3]

The simplest way to solve both these problems at once is to introduce *two* Higgs doublets. Instead of H, we thus introduce two chiral multiplets H_u and H_d.

[3] Assigning instead the quantum numbers $(\mathbf{1},\mathbf{2})_{-1/2}$ to the Higgs multiplet, the couplings in Eq. (2.62) would be allowed (after exchanging $h^\dagger \leftrightarrow h$), but then those of Eq. (2.61) would be forbidden.

Higgs fields and higgsinos:

spin-	0	1/2	multiplet	SU(3)	SU(2)	U(1)
	h_u	$\tilde{h}_u$	H_u	**1**	**2**	1/2
	h_d	$\tilde{h}_d$	H_d	**1**	**2**	−1/2

The gauge anomalies cancel between $\tilde{h}_u$ and $\tilde{h}_d$, since we have added a pair of Weyl spinors in conjugate representations to the already anomaly-free matter content of the Standard Model. Furthermore, the superpotential

$$W = y^{(u)}_{IJ}\, H_u Q_I U_J + y^{(d)}_{IJ}\, H_d Q_I D_J + y^{(e)}_{IJ}\, H_d L_I E_J \tag{2.63}$$

contains Yukawa couplings for both the up-type quarks and the down-type quarks and leptons, and can therefore induce masses for all Standard Model fermions provided that both h_u and h_d acquire vacuum expectation values. The details of electroweak symmetry breaking will be investigated in the following Section.

The most general gauge-invariant renormalizable superpotential for the MSSM fields reads

$$\begin{aligned} W_{\text{MSSM}} = {}& y^{(u)}_{IJ}\, H_u Q_I U_J + y^{(d)}_{IJ}\, H_d Q_I D_J + y^{(e)}_{IJ}\, H_d L_I E_J + \mu\, H_u H_d \\ & + \lambda_{IJK}\, L_I L_J E_K + \lambda'_{IJK}\, L_I Q_J D_K + \lambda''_{IJK}\, U_I D_J D_K + \mu'_I\, H_u L_I\,. \end{aligned} \tag{2.64}$$

The first line contains, besides the Yukawa terms above, a supersymmetric Higgs mass term μ. The interactions in the second line are problematic: In components, they lead to Yukawa-like couplings between squarks, quarks, sleptons and leptons, and to mass mixing between leptons and higgsinos. All these terms violate baryon or lepton number. Baryon- and lepton-number violating interactions are extremely constrained by experiment. For instance, if the squark masses were $\mathcal{O}(1\text{ TeV})$ and the λ' and λ'' couplings were $\mathcal{O}(1)$, then squarks would mediate extremely fast proton decay with a predicted proton lifetime of less than a second (while the observed limit is $\gtrsim 10^{32}$ years). It is therefore commonly assumed that the λ, λ', λ'' and μ' couplings are forbidden by an additional discrete symmetry, the simplest example of which is called *matter parity.* Matter parity assigns a $\mathbb{Z}_2$ charge of -1 to all matter chiral superfields, and a charge of $+1$ to both Higgs superfields. Expressed in terms of baryon and lepton number,

$$P_M = (-1)^{3(B-L)}\,. \tag{2.65}$$

An equivalent symmetry is *R-parity*, which acts on component fields as

$$P_R = (-1)^{3(B-L)+2s} \tag{2.66}$$

where s is the spin. In terms of R-parity, all Standard Model fields have charge $+1$ and all their superpartners have charge -1.

While it can be interesting to study (small) R-parity violation, for the remainder of this section we will assume that R-parity is exact. The superpotential is then simply

$$W_{\text{MSSM}} = y^{(u)}_{IJ}\, H_u Q_I U_J + y^{(d)}_{IJ}\, H_d Q_I D_J + y^{(e)}_{IJ}\, H_d L_I E_J + \mu\, H_u H_d\,. \tag{2.67}$$

Apart from preventing baryon- and lepton-number violating processes, R-parity has some more interesting implications for phenomenology; more on this later.

Besides the supersymmetric interactions derived from Eq. (2.67), we also need to give the supersymmetry-breaking terms in the Lagrangian. As announced in the previous Section, we will not specify any model of spontaneous supersymmetry breaking, but rather parameterize the effects of SUSY breaking on the MSSM fields by soft SUSY-breaking terms. The complete list of soft terms allowed by gauge symmetry and R-parity is

$$\begin{aligned}\mathcal{L}_{\text{soft}} = &-\frac{1}{2}\sum_{a=1}^{3} M_a \,\text{tr}\,\lambda_a\lambda_a + \text{h.c.}\\ &- m^2_{QIJ}\,\tilde{q}_I^\dagger\tilde{q}_J - m^2_{UIJ}\,\tilde{u}_I^\dagger\tilde{u}_J - m^2_{DIJ}\,\tilde{d}_I^\dagger\tilde{d}_J - m^2_{LIJ}\,\tilde{\ell}_I^\dagger\tilde{\ell}_J - m^2_{EIJ}\,\tilde{e}_I^\dagger\tilde{e}_J\\ &- a_{UIJ}\,h_u\tilde{q}_I\tilde{u}_J - a_{DIJ}\,h_d\tilde{q}_I\tilde{d}_J - a_{EIJ}\,h_d\tilde{\ell}_I\tilde{e}_J + \text{h.c.}\\ &- m^2_{H_u}\,|h_u|^2 - m^2_{H_d}\,|h_d|^2 - (b\,h_u h_d + \text{h.c.})\ .\end{aligned} \tag{2.68}$$

Here M_a are complex gaugino masses, $m^2_{Q,U,D,L,E}$ are hermitian squark and slepton mass matrices, $a_{U,D,E}$ are general complex matrices of trilinear scalar couplings, $m^2_{H_u}$ and $m^2_{H_d}$ are real mass parameters for the up-type and down-type Higgs fields, and b is a complex mass mixing parameter for the Higgs scalars.

One of the problems of parameterizing SUSY breaking in the most general way is that the number of parameters is huge. With unbroken supersymmetry, the MSSM has one free parameter less than the Standard Model: in both cases the gauge couplings and the Yukawa couplings (modulo flavour rotations) are free parameters, as is the Higgs mass, but the Higgs self-coupling in the MSSM is fixed by supersymmetry. This changes drastically when taking SUSY breaking into account, and when parameterizing SUSY breaking by Eq. (2.68). Careful counting reveals that the soft terms in Eq. (2.68) contain 105 physical real parameters (most of these are actually CP-violating phases and flavour-changing mass mixings, which are tightly constrained by experiment). Any manageable phenomenological analysis must make some additional assumptions, or specify a model for generating the soft terms.

2.2.3 Electroweak symmetry breaking in the MSSM

We proceed to analyze the MSSM Higgs sector in some more detail. The MSSM contains two scalar Higgs doublets,

$$h_u = \begin{pmatrix} h_u^+ \\ h_u^0 \end{pmatrix}, \qquad h_d = \begin{pmatrix} h_d^0 \\ h_d^- \end{pmatrix}. \tag{2.69}$$

Here the superscripts of the doublet components denote the electric charge Q (recall that $Q = Y + T_3$, where Y is the hypercharge with $Y[h_u] = +\frac{1}{2}$ and $Y[h_d] = -\frac{1}{2}$, and T_3 is the weak isospin eigenvalue). By a choice of gauge, we can set h_u^+ to zero:

$$h_u = \begin{pmatrix} 0 \\ h_u^0 \end{pmatrix}. \tag{2.70}$$

At the minimum of the potential, electric charge is unbroken, so also h_d^- vanishes:

$$\langle h_d \rangle = \begin{pmatrix} \langle h_d^0 \rangle \\ 0 \end{pmatrix}. \tag{2.71}$$

The μ term in the superpotential Eq. (2.67) gives rise to supersymmetric mass terms for h_u^0 and h_d^0. In addition, there are soft mass parameters $m_{H_u}^2$ and $m_{H_d}^2$ in Eq. (2.68),[4] as well as a soft mass mixing parameter b. Finally, there is a quartic Higgs self-interaction from the SU(2) × U(1) D-term potential. Altogether the scalar potential for h_u^0 and h_d^0 is, when setting the squarks and sleptons to zero (→ Exercise)

$$\begin{aligned}\mathcal{V} = &\left(|\mu|^2 + m_{H_u}^2\right) |h_u^0|^2 + \left(|\mu|^2 + m_{H_d}^2\right) |h_d^0|^2 + \left(b\, h_u^0 h_d^0 + \text{h.c.}\right) \\ &+ \frac{1}{8}\left(g^2 + g'^2\right)\left(|h_u^0|^2 - |h_d^0|^2\right)^2 .\end{aligned} \tag{2.72}$$

Since b is the only parameter which depends on the phases of h_u^0 and h_d^0, we can absorb any phase by a field redefinition into the Higgs fields and choose b real and positive, $b > 0$, without loss of generality. Two conditions for obtaining a stable potential which breaks electroweak symmetry are (→ Exercise)

$$\left(m_{H_u}^2 + |\mu|^2\right)\left(m_{H_d}^2 + |\mu|^2\right) - b^2 < 0\,, \qquad m_{H_u}^2 + m_{H_d}^2 + 2\,|\mu|^2 - 2\,b > 0\,. \tag{2.73}$$

If these conditions are satisfied, the Higgs fields should acquire vacuum expectation values $\langle h_u^0 \rangle = v_u$ and $\langle h_d^0 \rangle = v_d$, breaking electroweak symmetry at a scale

$$v^2 = v_u^2 + v_d^2 = (174\,\text{GeV})^2 = \frac{(246\,\text{GeV})^2}{2}\,. \tag{2.74}$$

As in the Standard Model, this gives masses to three of the electroweak gauge bosons, while the photon remains massless:

$$m_{W^\pm} = \frac{g}{\sqrt{2}}\, v = 80\ \text{GeV}\,, \qquad m_Z = \sqrt{\frac{g^2 + g'^2}{2}}\, v = 91\ \text{GeV}\,, \qquad m_\gamma = 0\,. \tag{2.75}$$

Defining an angle β by $\tan\beta = \frac{v_u}{v_d}$, the vacuum expectation values are related to the fundamental parameters in the Higgs potential as

$$\frac{\sin 2\beta}{2} = \frac{1}{\tan\beta + \cot\beta} = \frac{b}{m_{H_u}^2 + m_{H_d}^2 + 2\,|\mu|^2} \tag{2.76}$$

and

$$m_Z^2 = \frac{\left|m_{H_d}^2 - m_{H_u}^2\right|}{|\cos 2\beta|} - m_{H_u}^2 - m_{H_d}^2 - 2\,|\mu|^2\,. \tag{2.77}$$

At large $\tan\beta$, we obtain the approximate relation (assuming $m_{H_u}^2 < m_{H_d}^2$)

$$m_Z^2 \approx -2\left(m_{H_u}^2 + |\mu|^2\right) \tag{2.78}$$

which explicitly shows that the Higgs mass parameter $m_{H_u}^2$ should be negative in a large part of parameter space, in order to trigger electroweak symmetry breaking.

In the Standard Model, the Higgs field contains four real degrees of freedom (from two complex doublets), three of which are eaten by the gauge bosons to furnish the longditudinal modes of the $W^\pm$ and Z. The remaining real scalar degree of freedom is the

[4] Despite the suggestive notation, the parameters $m_{H_u}^2$ and $m_{H_d}^2$ need not be positive. Indeed, in generic models of SUSY breaking, the mass parameter $m_{H_u}^2$ is driven to negative values at the electroweak scale by radiative corrections caused by the large top Yukawa coupling. The fact that electroweak symmetry breaking is thus radiatively induced, rather than just the consequence of an arbitrary parameter choice as in the Standard Model, is also sometimes advocated as an advantage of supersymmetry.

famous Higgs boson. In the MSSM, we have eight real degrees of freedom, which after electroweak symmetry breaking leaves five physical Higgs bosons. These are two CP-even neutral scalars h^0 and H^0, one CP-odd pseudoscalar A^0, and a complex charged scalar $H^\pm$. They are contained in the fluctuations about the symmetry-breaking vacuum as

$$\begin{pmatrix} h_u^0 \\ h_d^0 \end{pmatrix} = \begin{pmatrix} v_u \\ v_d \end{pmatrix} + \frac{1}{\sqrt{2}} R_\alpha \begin{pmatrix} h_0 \\ H_0 \end{pmatrix} + \frac{i}{\sqrt{2}} R_{\beta_0} \begin{pmatrix} G^0 \\ A^0 \end{pmatrix} \tag{2.79}$$

and

$$\begin{pmatrix} h_u^+ \\ h_d^{-*} \end{pmatrix} = R_{\beta_\pm} \begin{pmatrix} G^+ \\ H^+ \end{pmatrix} . \tag{2.80}$$

Here G^0 and G^+ are the "would-be Goldstone" components of the Z and W^+ boson. R_α, R_{β_0} and $R_{\beta_\pm}$ are mixing matrices:

$$\begin{gathered} R_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} , \\ R_{\beta_0} = \begin{pmatrix} \sin\beta_0 & \cos\beta_0 \\ -\cos\beta_0 & \sin\beta_0 \end{pmatrix} , \quad R_{\beta_\pm} = \begin{pmatrix} \sin\beta_p m & \cos\beta_\pm \\ -\cos\beta_\pm & \sin\beta_\pm \end{pmatrix} \end{gathered} \tag{2.81}$$

which should be chosen such that the mass matrix is diagonal, i.e.

$$\mathcal{V} = \frac{1}{2} m_{h^0}^2 (h^0)^2 + \frac{1}{2} m_{H^0}^2 (H^0)^2 + \frac{1}{2} m_{A^0}^2 (A^0)^2 + m_{H^\pm}^2 |H^\pm|^2 + \dots \tag{2.82}$$

with the Goldstone masses vanishing. This implies

$$\beta_0 = \beta_\pm = \beta . \tag{2.83}$$

The mass eigenvalues are

$$\begin{aligned} m_{A^0}^2 &= \frac{2\,b}{\sin 2\beta} = 2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2 , \\ m_{h^0,H^0}^2 &= \frac{1}{2}\left(m_{A^0}^2 + m_Z^2 \mp \sqrt{\left(m_{A^0}^2 - m_Z^2\right)^2 + 4\, m_Z^2 m_{A^0}^2 \sin^2 2\beta} \right) , \\ m_{H^\pm}^2 &= m_{A^0}^2 + m_W^2 . \end{aligned} \tag{2.84}$$

Finally, the neutral Higgs boson mixing angle α is conventionally chosen negative. It is related to the masses by

$$\frac{\sin 2\alpha}{\sin 2\beta} = -\frac{m_{H^0}^2 + m_{h^0}^2}{m_{H_0}^2 - m_{h^0}^2} , \qquad \frac{\tan 2\alpha}{\tan 2\beta} = \frac{m_{A^0}^2 + m_Z^2}{m_{A_0}^2 - m_Z^2} . \tag{2.85}$$

A frequently considered limit is the *decoupling limit* in which $m_{A^0} \gg m_Z$. In this limit the heavier Higgs particles A^0, H^0 and $H^\pm$ effectively decouple, and one is left with a single Standard Model-like Higgs h^0. The mixing angle α then approaches

$$\alpha \to \beta - \frac{\pi}{2} \tag{2.86}$$

and

$$m_{H^\pm}^2 \to m_{A^0}^2 , \quad m_{H^0}^2 \to m_{A^0}^2 , \quad m_{h^0}^2 \to m_Z^2 \cos^2 2\beta . \tag{2.87}$$

Notice the inequality

$$m_{h^0} < m_Z \tag{2.88}$$

(which actually holds more generally, not just in the decoupling limit). This at first sight seems like a disastrous contradiction with the data, which currently favours a Standard Model-like Higgs at a mass $m_{h^0} = 125$ GeV.[5] However, it should be noted that our formulas for the Higgs masses hold only at the tree level. Loop corrections can significantly affect the Higgs masses, and in particular can lift m_{h^0} from its tree-level bound $m_Z = 91$ GeV to the desired 125 GeV.

Let us superficially sketch a derivation of the leading loop correction to the lightest Higgs mass in the MSSM, which is due to the top-stop sector, i.e. the third generation up-type quarks and squarks. They acccount for the dominant effect simply because the top and stops are coupled to the Higgs fields by a large Yukawa coupling. The diagrams we need to calculate are the same as in Section 2.2.1, with electrons and selectrons replaced by tops and stops. There is however a quicker way to obtain the one-loop corrected Higgs mass in a certain limit, by using the Coleman-Weinberg formula for the one-loop effective potential:

$$\mathcal{V}_{\rm CW} = \frac{1}{64\pi^2}\,{\rm Str}\,|\mathcal{M}|^4\left(\log\frac{|\mathcal{M}|^2}{Q^2} - \delta\right). \tag{2.89}$$

In this formula, "Str" denotes a trace over all particles involved, with a minus sign for fermions; $\mathcal{M}$ is the effective (field-dependent) mass matrix; Q is the renormalization scale; and δ is a renormalization scheme-dependent constant (which may be absorbed into a redefinition of Q). For the top-stop contribution, the relevant mass matrices are that of the top quark, whose mass is solely due to electroweak symmetry breaking,

$$|\mathcal{M}_t|^2 = \begin{pmatrix} y_t^2|h_u^0|^2 & 0 \\ 0 & y_t^2|h_u^0|^2 \end{pmatrix}, \tag{2.90}$$

and that of the stop squarks, which also includes the appropriate soft masses:

$$|\mathcal{M}_{\tilde{t}}|^2 = \begin{pmatrix} m_{Q33}^2 + y_t^2|h_u^0|^2 & a_{U33}h_u^0 + y_t\mu h_d^0 \\ a_{U33}^* h_u^{0*} + y_t\mu^* h_d^{0*} & m_{U33}^2 + |y_t|^2|h_u^0|^2 \end{pmatrix}. \tag{2.91}$$

Here the subdominant D-term potential contributions have been ignored (since they do not involve y_t but the much smaller electroweak gauge couplings). The resulting Higgs mass correction becomes especially simple when taking simultaneously the large $\tan\beta$ limit, the decoupling limit, and the limit of degenerate left-handed and right-handed squark soft masses $m_{Q33}^2 = m_{U33}^2$. One can then simply set $h_u^0 = v_u + h^0/\sqrt{2}$, and extract the terms proportional to ${h^0}^2$ in Eq. (2.89). The renormalization scale is fixed by the condition that the one-loop tadpole (linear) term should vanish. This eventually gives

$$\delta m_{h^0}^2 = \frac{3}{4\pi^2}y_t^2 m_t^2\left(\log\frac{m_{\tilde{t}_1}m_{\tilde{t}_2}}{m_t^2} + \frac{A_t^2}{m_{\tilde{t}_1}m_{\tilde{t}_2}}\left(1 - \frac{A_t^2}{12\,m_{\tilde{t}_1}m_{\tilde{t}_2}}\right)\right). \tag{2.92}$$

Here $A_t \equiv a_{U33}/y_t$, and the $m_{\tilde{t}_i}$ are the roots of the eigenvalues of the stop mass matrix. The parameters in Eq. (2.92) are running parameters, so the approximation depends on the scales where they are evaluated, but the uncertainty due to this scale variation is of the same order as the higher-order loop corrections. Modern computer codes can calculate radiative corrections to the Higgs mass in far more general situations (at any $\tan\beta$, away from decoupling, and for non-degenerate m_{Q33}^2 and m_{U33}^2), incorporating also

[5] Note however that, at the time of writing, there remains a tiny region of parameter space in which the 125 GeV Higgs boson is interpreted as the H^0, while the lighter h^0 has escaped detection because it has reduced couplings to Standard Model states.

the effects from other fields besides the top and stops, and up to three-loop precision. However, much can be learned already from the simplified result Eq. (2.92).

The main lesson is perhaps that, in order to obtain a physical Higgs mass of 125 GeV, the loop corrections need to be very sizeable. Even assuming that the tree-level Higgs mass saturates at m_Z, we need $\delta m^2_{h^0} \approx (86\,\mathrm{GeV})^2$ which is nearly as large as the tree-level value. From Eq. (2.92) it is evident that this either requires very heavy stops, with $m_{\tilde{t}_i}$ of the order of several TeV, or a sizeable stop mixing parameter A_t. This, besides the fact that no supersymmetric particles have been discovered so far, is often held against the MSSM as a plausible theory of Nature. It is sometimes called the *little hierarchy problem*.

The problem is essentially the following: To solve the "big" electroweak hierarchy problem of quadratic divergences, one would expect the superpartner mass scale to be close to the electroweak scale. On the other hand, to obtain the correct Higgs mass, the superpartner mass scale seems to be at least an order of magnitude higher. Obtaining both large loop corrections to the Higgs mass and the proper electroweak scale then requires inexplicable cancellations between a priori unrelated parameters. For instance, in Eq. (2.78) $|m^2_{H_u}|$ and $|\mu|^2$ must either both be of the order of m_Z^2, or (if they are significantly larger) they must be finely tuned to cancel each other to great precision. On the other hand, if the stop masses are around a few TeV, then this is also the characteristic size of $|m^2_{H_u}|$ (a hierarchy between the Higgs and the stop soft masses would not be stable under radiative corrections, so they are generically of comparable magnitude). It therefore appears that the MSSM can be a realistic model only with substantial fine-tuning. The little hierarchy problem can be alleviated in non-minimal supersymmetric extensions of the Standard Model, in which the tree-level bound Eq. (2.88) on the Higgs mass is relaxed.

Exercise 18: Higgs potential in the MSSM

a. Compute the scalar potential for the Higgs fields $h_u = (h_u^+, h_u^0)$ and $h_d = (h_d^0, h_d^-)$ in the MSSM (setting all squark and slepton fields to zero). Thus, verify Eq. (2.72). Give an argument why b can always be chosen real and non-negative.

b. The Higgs potential should be bounded from below. Show that this requires

$$m^2_{H_u} + m^2_{H_d} + 2|\mu|^2 - 2\,b > 0\,.$$

c. The point $h_u = h_d = 0$, which preserves electroweak symmetry, should be unstable. Show that this requires

$$(m^2_{H_u} + |\mu|^2)(m^2_{H_d} + |\mu|^2) - b^2 < 0\,.$$