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Supersymmetry and Extra Dimensions

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Abstract: These are lecture notes for the Cambridge mathematics tripos Part III Supersymmetry course, based on Ref. [1]. You should have attended the required courses: Quantum Field Theory, and Symmetries and Particle Physics. You will find the latter parts of Advanced Quantum Field theory (on renormalisation) useful. The Standard Model course will aid you with one topic (the minimal supersymmetric standard model), and help with understanding spontaneous symmetry breaking. These lecture notes, and the four accompanying examples sheets may be found on the DAMTP pages, and there will be classes by M.Nardecchia@damtp.cam.ac.uk organised for each examples sheet. You can watch videos of the lectures on the web by following the links from

`http://users.hepforge.org/~allanach/teaching.html`
`http://www.damtp.cam.ac.uk/user/fq201/`

I have a tendency to make trivial transcription errors on the board - please stop me if I make one.

In general, the books contain several typographical errors. The last two books on the list have a different metric convention to the one used herein (switching metric conventions is surprisingly irksome!)

Books

- Bailin and Love, "Supersymmetric gauge field theory and string theory", Institute of Physics publishing has nice explanations.
- Lykken "Introduction to supersymmetry", `arXiv:hep-th/9612114` - particularly good on extended supersymmetry.
- Aithchison, "Supersymmetry in particle physics", Cambridge University Press is super clear and basic.
- Martin "A supersymmetry primer", `arXiv:hep-ph/9709356` a detailed and phenomenological reference.
- Wess and Bagger, "Supersymmetry and Supergravity", Princeton University Publishing is terse but has no errors that I know of.

We welcome questions during lectures.

Contents

1 Physical Motivation

Let us review some relevant facts about the universe we live in.

1.1 Basic theory: QFT

Microscopically we have *quantum mechanics* and *special relativity* as two fundamental theories.
A consistent framework incorporating these two theories is *quantum field theory (QFT)*. In this theory the fundamental entities are quantum fields. Their excitations correspond to the physically observable elementary particles which are the basic constituents of matter

as well as the mediators of all the known interactions. Therefore, fields have a particle-like character. Particles can be classified in two general classes: bosons (spin $s = n \in \mathbb{Z}$) and fermions ($s = n + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}$). Bosons and fermions have very different physical behaviour. The main difference is that fermions can be shown to satisfy the PAULI "exclusion principle" , which states that two identical fermions cannot occupy the same quantum state, and therefore explaining the vast diversity of atoms.
All elementary matter particles: the leptons (including electrons and neutrinos) and quarks (that make protons, neutrons and all other hadrons) are fermions. Bosons on the other hand include the photon (particle of light and mediator of electromagnetic interaction), and the mediators of all the other interactions. They are not constrained by the Pauli principle. As we will see, *supersymmetry* is a symmetry that unifies bosons and fermions despite all their differences.

1.2 Basic principle: symmetry

If QFT is the basic framework to study elementary processes, one tool to learn about these processes is the concept of *symmetry*.
A symmetry is a transformation that can be made to a physical system leaving the physical observables unchanged. Throughout the history of science symmetry has played a very important role to better understand nature.

1.3 Classes of symmetries

For elementary particles, we can define two general classes of symmetries:

- *Space-time symmetries:* These symmetries correspond to transformations on a field theory acting explicitly on the space-time coordinates,

  $$x^\mu \quad \mapsto \quad x'^\mu\,(x^\nu) \ , \qquad \mu,\nu = 0,1,2,3 \ .$$

  Examples are rotations, translations and, more generally, *Lorentz- and Poincaré transformations* defining special relativity as well as *general coordinate transformations* that define *general relativity*.

- *Internal symmetries:* These are symmetries that correspond to transformations of the different fields in a field theory,

  $$\Phi^a(x) \quad \mapsto \quad M^a{}_b\,\Phi^b(x) \ .$$

  Roman indices a,b label the corresponding fields. If $M^a{}_b$ is constant then the symmetry is a *global symmetry*; in case of space-time dependent $M^a{}_b(x)$ the symmetry is called a *local symmetry*.

1.4 Importance of symmetries

Symmetry is important for various reasons:

- *Labelling and classifying particles:* Symmetries label and classify particles according to the different conserved quantum numbers identified by the space-time and internal symmetries (mass, spin, charge, colour, etc.). In this regard symmetries actually "define" an elementary particle according to the behaviour of the corresponding field with respect to the different symmetries.

- Symmetries determine the *interactions* among particles, by means of the *gauge principle*, for instance. It is important that *most QFTs of vector bosons are sick: they are non-renormalisable.* The counter example to this is *gauge* theory, where vector bosons are *necessarily in the adjoint representation* of the gauge group. As an illustration, consider the Lagrangian

$$\mathcal{L} \;=\; \partial_\mu \phi \, \partial^\mu \phi^* \;-\; V(\phi, \phi^*)$$

which is invariant under rotation in the complex plane

$$\phi \;\mapsto\; \exp(i\alpha)\, \phi \;,$$

as long as α is a constant (global symmetry). If $\alpha = \alpha(x)$, the kinetic term is no longer invariant:

$$\partial_\mu \phi \;\mapsto\; \exp(i\alpha) \left(\partial_\mu \phi \,+\, i(\partial_\mu \alpha)\phi\right).$$

However, the covariant derivative D_μ, defined as

$$D_\mu \phi \;:=\; \partial_\mu \phi \;+\; i A_\mu \, \phi \;,$$

transforms like ϕ itself, if the gauge - potential A_μ transforms to $A_\mu - \partial_\mu \alpha$:

$$D_\mu \phi \;\mapsto\; \exp(i\alpha) \left(\partial_\mu \phi \,+\, i(\partial_\mu \alpha)\phi \;+\; i(A_\mu - \partial_\mu \alpha)\, \phi\right) \;=\; \exp(i\alpha)\, D_\mu \phi \;,$$

so we rewrite the Lagrangian to ensure gauge - invariance:

$$\mathcal{L} \;=\; D_\mu \phi \, D^\mu \phi^* \;-\; V(\phi, \phi^*) \,.$$

The scalar field ϕ couples to the gauge - field A_μ via $A_\mu \phi A^\mu \phi$, similarly, the Dirac Lagrangian

$$\mathcal{L} \;=\; \overline{\Psi}\, \gamma^\mu \, D_\mu \Psi$$

has an interaction term $\overline{\Psi} A_\mu \Psi$. This interaction provides the three point vertex that describes interactions of electrons and photons and illustrate how photons mediate the electromagnetic interactions.

- Symmetries can hide or be *spontaneously broken:* Consider the potential $V(\phi, \phi^*)$ in the scalar field Lagrangian above.

  If $V(\phi, \phi^*) = V(|\phi|^2)$, then it is symmetric for $\phi \mapsto \exp(i\alpha)\phi$. If the potential is of the type

$$V \;=\; a\, |\phi|^2 \;+\; b\, |\phi|^4 \;, \qquad a, b \geq 0 \;,$$

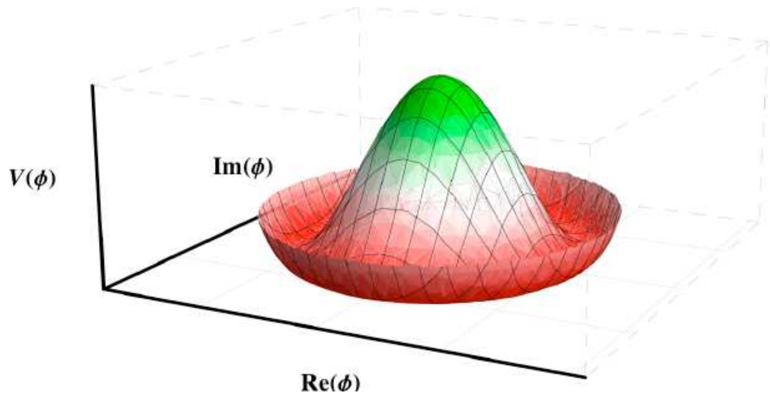


Figure 1. The Mexican hat potential for $V = \left(a - b\,|\phi|^2\right)^2$ with $a, b \geq 0$.

then the minimum is at $\langle\phi\rangle = 0$ (here $\langle\phi\rangle \equiv \langle 0|\phi|0\rangle$ denotes the *vacuum expectation value (VEV)* of the field ϕ). The vacuum state is then also symmetric under the symmetry since the origin is invariant. However if the potential is of the form

$$V = \left(a - b\,|\phi|^2\right)^2, \qquad a, b \geq 0\ ,$$

the symmetry of V is lost in the ground state $\langle\phi\rangle \neq 0$. The existence of hidden symmetries is important for at least two reasons:

(i) This is a natural way to introduce an energy scale in the system, determined by the non vanishing VEV. In particular, we will see that for the standard model $M_{\rm ew} \approx 10^3$ GeV, defines the basic scale of mass for the particles of the standard model, the electroweak gauge bosons and the matter fields, through their Yukawa couplings, obtain their mass from this effect.

(ii) The existence of hidden symmetries implies that the fundamental symmetries of nature may be huge despite the fact that we observe a limited amount of symmetry. This is because the only manifest symmetries we can observe are the symmetries of the vacuum we live in and not those of the full underlying theory. This opens-up an essentially unlimited resource to consider physical theories with an indefinite number of symmetries even though they are not explicitly realised in nature. The standard model is the typical example and supersymmetry and theories of extra dimensions are further examples.

1.4.1 The Standard Model

The Standard Model is well-defined and currently well confirmed by experiments.

- space-time symmetries: Poincaré in 4 dimensions
- gauged G_{SM}=SU(3)$_c\times$SU(2)$_L\times$U(1)$_Y$ symmetry, where SU(3)$_c$ defines the strong interactions. SU(2)$_L\times$U(1)$_Y$ is spontaneously broken by the *Higgs* mechanism to U(1)$_{em}$. The gauge fields are spin-1 bosons, for example the photon A^μ, or gluons

$G^{a=1,\dots,8}$. Matter fields (quarks and leptons) have spin 1/2 and come in three 'families' (successively heavier copies). The Higgs boson (a particle has just been discovered at the LHC whose properties are consistent with the Higgs boson) is the spin zero particle that spontaneously breaks the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. The $W^\pm$ and Z particles get a mass via the Higgs mechanism and therefore the weak interactions are short range. This is also the source of masses for all quarks and leptons. The sub-index L in $SU(2)_L$ refers to the fact that the Standard Model does not preserve parity and differentiates between left-handed and right-handed particles. In the Standard Model only left-handed particles transform non-trivially under $SU(2)_L$. The gauge particles have all spin $s = 1\hbar$ and mediate each of the three forces: photons (γ) for $U(1)$ electromagnetism, gluons for $SU(3)_C$ of strong interactions, and the massive $W^\pm$ and Z for the weak interactions.

1.5 Problems of the Standard Model

The Standard Model is one of the cornerstones of all science and one of the great triumphs of the past century. It has been carefully experimentally verified in many ways, especially during the past 20 years. However, there are still some unresolved issues or mysteries:

- The hierarchy problem. The Higgs mass is $m_h \approx 126$ GeV, whereas the gravitational scale is $M_{Planck} \sim \sqrt{G} \sim 10^{19}$ GeV. The 'hierarchy problem' is: why is $m_h/M_{Planck} \sim 10^{-17}$ so much smaller than 1? In a fundamental theory, one might expect them to be the same order. In QFT, one sees that quantum corrections (loops) to v are expected to be of order of the heaviest scale in the theory divided by $16\pi^2$. The question of why the hierarchy is stable with respect to the quantum corrections is called the *technical hierarchy problem*, and is arguably the main motivation for weak-scale supersymmetry.

- The cosmological constant (Λ) problem: probably the biggest problem in fundamental physics. Λ is the energy density of free space time. Why is $(\Lambda/M_{Planck})^4 \sim 10^{-120} \ll 1$?

- The Standard Model has around 20 parameters, which must be measured then set 'by hand'.

- What particle constitutes the dark matter observed in the universe? It is not contained in the Standard Model.

We wish to find extensions that could solve some or all of the problems mentioned above in order to generalise the Standard Model. See the Part III Standard Model course for more details. Experiments are a traditional way of making progress in science. We need experiments to explore energies above the currently attainable scales and discover new particles and underlying principles that generalise the Standard Model. This approach is of course being followed at the LHC. The experiment will explore physics at the 10^3 GeV scale and new physics beyond the Standard Model. Notice that directly exploring energies closer to the Planck scale $M_{Planck} \approx 10^{19}$ GeV is out of the reach for many years to come.

1.5.1 Modifications of the Standard Model

In order to go beyond the Standard Model we can follow several avenues, for example:

- Add new particles and/or interactions (e.g. a dark matter particle).
- More symmetries. For example,

(i) internal symmetries, for example *grand unified theories (GUTs)* in which the symmetries of the Standard Model are themselves the result of the breaking of a yet larger symmetry group.

$$G_{\mathrm{GUT}} \quad \stackrel{M\approx 10^{16}\mathrm{GeV}}{\longrightarrow} \quad G_{\mathrm{SM}} \quad \stackrel{M\approx 10^{2}\mathrm{GeV}}{\longrightarrow} \quad SU(3)_c \times U(1)_Y ,$$

This proposal is very elegant because it unifies, in one single symmetry, the three gauge interactions of the Standard Model. It leaves unanswered most of the open questions above, except for the fact that it reduces the number of independent parameters due to the fact that there is only one gauge coupling at large energies. This is expected to "run" at low energies and give rise to the three different couplings of the Standard Model (one corresponding to each group factor). Unfortunately, with our present precision understanding of the gauge couplings and spectrum of the Standard Model, the running of the three gauge couplings does **not** unify at a single coupling at higher energies but they cross each other at different energies.

(ii) *Supersymmetry.* Supersymmetry is an external, or space-time symmetry. Supersymmetry solves the technical hierarchy problem due to cancellations between the contributions of bosons and fermions to the electroweak scale, defined by the Higgs mass. Combined with the GUT idea, it also solves the unification of the three gauge couplings at one single point at larger energies. Supersymmetry also provides the most studied example for dark matter candidates. Moreover, it provides well defined QFTs in which issues of strong coupling can be better studied than in the non-supersymmetric models.

(iii) Extra spatial dimensions. More general space-time symmetries open up many more interesting avenues. These can be of two types. First we can add more dimensions to space-time, therefore the Poincaré symmetries of the Standard Model and more generally the general coordinate transformations of general relativity, become substantially enhanced. This is the well known *Kaluza Klein theory* in which our observation of a 4 dimensional universe is only due to the fact that we have limitations about "seeing" other dimensions of space-time that may be hidden to our experiments. In recent years this has been extended to the *brane world scenario* in which our 4 dimensional universe is only a brane or surface inside a larger dimensional universe. These ideas may lead to a different perspective of the hierarchy problem and also may help unify internal and space-time symmetries.

- Beyond QFT: A QFT with Supersymmetry and extra dimensions does not address the problem of quantising gravity. For this purpose, the current best hope is string theory which goes beyond our basic framework of QFT. It so happens that for its consistency, string theory requires supersymmetry and extra dimensions also. This gives a further motivation to study supersymmetry.

2 Supersymmetry algebra and representations

2.1 Poincaré symmetry and spinors

The Poincaré group corresponds to the basic symmetries of special relativity, it acts on space-time coordinates x^μ as follows:

$$x^\mu \ \mapsto \ x'^\mu \ = \ \underbrace{\Lambda^\mu{}_\nu}_{\text{Lorentz}} \, x^\nu \ + \ \underbrace{a^\mu}_{\text{translation}}$$

Lorentz transformations leave the metric tensor $\eta_{\mu\nu} = \mathrm{diag}(1,\ -1,\ -1,\ -1)$ invariant:

$$\Lambda^T \, \eta \, \Lambda \ = \ \eta$$

They can be separated between those that are connected to the identity and those that are not (i.e. parity reversal $\Lambda_P = \mathrm{diag}(1,\ -1,\ -1,\ -1)$ and time reversal $\Lambda_T = \mathrm{diag}(-1,\ 1,\ 1,\ 1)$). We will mostly discuss those Λ continuously connected to identity, i.e. the *proper orthochronous group*[1] $SO(1,3)^\uparrow$. Generators for the Poincaré group are the hermitian $M^{\mu\nu}$ (rotations and Lorentz boosts) and P^σ (translations) with algebra

$$\begin{aligned}
\Big[P^\mu \ , \ P^\nu\Big] \ &= \ 0 \\
\Big[M^{\mu\nu} \ , \ P^\sigma\Big] \ &= \ i \left(P^\mu \, \eta^{\nu\sigma} \ - \ P^\nu \, \eta^{\mu\sigma}\right) \\
\Big[M^{\mu\nu} \ , \ M^{\rho\sigma}\Big] \ &= \ i \left(M^{\mu\sigma} \, \eta^{\nu\rho} \ + \ M^{\nu\rho} \, \eta^{\mu\sigma} \ - \ M^{\mu\rho} \, \eta^{\nu\sigma} \ - \ M^{\nu\sigma} \, \eta^{\mu\rho}\right)
\end{aligned}$$

A 4 dimensional matrix representation for the $M^{\mu\nu}$ is

$$(M^{\rho\sigma})^\mu{}_\nu \ = \ -i \left(\eta^{\mu\sigma} \, \delta^\rho{}_\nu \ - \ \eta^{\rho\mu} \, \delta^\sigma{}_\nu\right) \, .$$

2.1.1 Properties of the Lorentz group

We now show that locally (i.e. in terms of the algebra), we have a correspondence

$$SO(1,3) \ \cong \ SU(2) \times SU(2).$$

The generators of $SO(1,3)$ (J_i of rotations and K_i of Lorentz boosts) can be expressed as

$$J_i \ = \ \frac{1}{2} \, \epsilon_{ijk} \, M_{jk} \ , \qquad K_i \ = \ M_{0i} \ ,$$

and the Lorentz algebra written in terms of J's and K's is

$$[K_i,\ K_j] = -i\epsilon_{ijk} J_k, \qquad [J_i,\ K_j] = i\epsilon_{ijk} K_k, \qquad [J_i,\ J_j] = i\epsilon_{ijk} J_k.$$

[1] These consist of the subgroup of transformations which have $\det\Lambda = +1$ and $\Lambda^0_0 \geq 1$.

We now construct the linear[2] combinations (which are neither hermitian nor anti hermitian)

$$A_i \;=\; \frac{1}{2}\left(J_i + iK_i\right) \,, \qquad B_i \;=\; \frac{1}{2}\left(J_i - iK_i\right) \tag{2.1}$$

which satisfy $SU(2)\times SU(2)$ commutation relations

$$\left[A_i \,,\, A_j\right] \;=\; i\epsilon_{ijk}\,A_k \,, \qquad \left[B_i \,,\, B_j\right] \;=\; i\epsilon_{ijk}\,B_k \,, \qquad \left[A_i \,,\, B_j\right] \;=\; 0$$

Under parity $\hat{P}$, ($x^0 \mapsto x^0$ and $\vec{x} \mapsto -\vec{x}$) we have

$$J_i \;\mapsto\; J_i \,, \qquad K_i \;\mapsto\; -K_i \quad\Longrightarrow\quad A_i \;\leftrightarrow\; B_i \,.$$

We can interpret $\vec{J} = \vec{A} + \vec{B}$ as the physical spin.

On the other hand, there is a homeomorphism (not an isomorphism)

$$SO(1,3) \;\cong\; SL(2,\mathbb{C}) \,.$$

To see this, take a 4 vector X and a corresponding 2×2 - matrix $\tilde{x}$,

$$X \;=\; x_\mu\,e^\mu \;=\; (x_0 \,,\, x_1 \,,\, x_2 \,,\, x_3) \,, \qquad \tilde{x} \;=\; x_\mu\,\sigma^\mu \;=\; \begin{pmatrix} x_0+x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \,,$$

where σ^μ is the 4 vector of *Pauli matrices*

$$\sigma^\mu \;=\; \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \,.$$

Transformations $X \mapsto \Lambda X$ under $SO(1,3)$ leaves the square

$$|X|^2 \;=\; x_0^2 \,-\, x_1^2 \,-\, x_2^2 \,-\, x_3^2$$

invariant, whereas the action of $SL(2,\mathbb{C})$ mapping $\tilde{x} \mapsto N\tilde{x}N^\dagger$ with $N \in SL(2,\mathbb{C})$ preserves the determinant

$$\det \tilde{x} \;=\; x_0^2 \,-\, x_1^2 \,-\, x_2^2 \,-\, x_3^2 \,.$$

The map between $SL(2,\mathbb{C})$ and $SO(1,3)$ is 2-1, since $N = \pm\mathbb{1}$ both correspond to $\Lambda = \mathbb{1}$, but $SL(2,\mathbb{C})$ has the advantage of being simply connected, so $SL(2,\mathbb{C})$ is the universal covering group.

2.1.2 Representations and invariant tensors of $SL(2,\mathbb{C})$

The basic representations of $SL(2,\mathbb{C})$ are:

- The fundamental representation

$$\psi'_\alpha \;=\; N_\alpha{}^\beta\,\psi_\beta \,, \qquad \alpha,\beta = 1,2 \tag{2.2}$$

The elements of this representation ψ_α are called *left-handed Weyl spinors*.

[2] NB $\epsilon_{123} = +1 = \epsilon^{123}$.

- The conjugate representation

$$\bar{\chi}'_{\dot{\alpha}} = N^{*\ \dot{\beta}}_{\dot{\alpha}}\, \bar{\chi}_{\dot{\beta}} \ , \qquad \dot{\alpha}, \dot{\beta} = 1, 2$$

  Here $\bar{\chi}_{\dot{\beta}}$ are called *right-handed Weyl spinors.*

- The contravariant representations are

$$\psi'^{\alpha} = \psi^{\beta}\, (N^{-1})_{\beta}^{\ \alpha} \ , \qquad \bar{\chi}'^{\dot{\alpha}} = \bar{\chi}^{\dot{\beta}}\, (N^{*-1})_{\dot{\beta}}^{\ \dot{\alpha}}.$$

The fundamental and conjugate representations are the basic representations of $SL(2, \mathbb{C})$ and the Lorentz group, giving then the importance to spinors as the basic objects of special relativity, a fact that could be missed by not realising the connection of the Lorentz group and $SL(2, \mathbb{C})$. We will see next that the contravariant representations are however not independent.
To see this we will consider now the different ways to raise and lower indices.

- The metric tensor $\eta^{\mu\nu} = (\eta_{\mu\nu})^{-1}$ is invariant under $SO(1, 3)$ and is used to raise/lower indices.

- The analogy within $SL(2, \mathbb{C})$ is

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot{\alpha}\dot{\beta}} \ , \ \epsilon^{12} = +1, \epsilon^{21} = -1.$$

  since

$$\epsilon'_{\alpha\beta} = N_{\alpha}^{\ \rho}\, N_{\beta}^{\ \sigma}\, \epsilon_{\rho\sigma} = \epsilon_{\alpha\beta} \cdot \det N = \epsilon_{\alpha\beta} \ .$$

  That is why ϵ is used to raise and lower indices

$$\psi^{\alpha} = \epsilon^{\alpha\beta}\psi_{\beta} \ , \qquad \bar{\chi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{\dot{\beta}} \ \Rightarrow \ \psi_{\alpha} = \epsilon_{\alpha\beta}\psi^{\beta}, \qquad \bar{\chi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\beta}}$$

  so contravariant representations are not independent from covariant ones.

- To handle mixed $SO(1, 3)$- and $SL(2, \mathbb{C})$ indices, recall that the transformed components x_{μ} should look the same, whether we transform the vector X via $SO(1, 3)$ or the matrix $\tilde{x} = x_{\mu}\sigma^{\mu}$ via $SL(2, \mathbb{C})$

$$(x_{\mu}\, \sigma^{\mu})_{\alpha\dot{\alpha}} \ \mapsto \ N_{\alpha}^{\ \beta}\, (x_{\nu}\, \sigma^{\nu})_{\beta\dot{\gamma}}\, N^{*\ \dot{\gamma}}_{\dot{\alpha}} = \Lambda_{\mu}^{\ \nu}\, x_{\nu}\, (\sigma^{\mu})_{\alpha\dot{\alpha}} \ ,$$

  so the correct transformation rule is

$$(\sigma^{\mu})_{\alpha\dot{\alpha}} = N_{\alpha}^{\ \beta}\, (\sigma^{\nu})_{\beta\dot{\gamma}}\, (\Lambda)^{\mu}_{\ \nu}\, N^{*\ \dot{\gamma}}_{\dot{\alpha}} \ .$$

  Similar relations hold for

$$(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} := \epsilon^{\alpha\beta}\, \epsilon^{\dot{\alpha}\dot{\beta}}\, (\sigma^{\mu})_{\beta\dot{\beta}} = (\mathbb{1}, \ -\vec{\sigma}) \ .$$

2.1.3 Generators of $SL(2,\mathbb{C})$

Let us define tensors $\sigma^{\mu\nu}$, $\bar{\sigma}^{\mu\nu}$ as antisymmetrised products of σ matrices:

$$\begin{aligned}(\sigma^{\mu\nu})_\alpha^{\ \beta} &:= \frac{i}{4}\left(\sigma^\mu\,\bar{\sigma}^\nu \;-\; \sigma^\nu\,\bar{\sigma}^\mu\right)_\alpha^{\ \beta} \\ (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} &:= \frac{i}{4}\left(\bar{\sigma}^\mu\,\sigma^\nu \;-\; \bar{\sigma}^\nu\,\sigma^\mu\right)^{\dot{\alpha}}_{\ \dot{\beta}}\end{aligned}$$

which satisfy the Lorentz algebra

$$\left[\sigma^{\mu\nu}\ ,\ \sigma^{\lambda\rho}\right] \;=\; i\left(\eta^{\mu\rho}\,\sigma^{\nu\lambda} \;+\; \eta^{\nu\lambda}\,\sigma^{\mu\rho} \;-\; \eta^{\mu\lambda}\,\sigma^{\nu\rho} \;-\; \eta^{\nu\rho}\,\sigma^{\mu\lambda}\right),$$

and analagously for $\bar{\sigma}^{\mu\nu}$. They thus form representations of the Lorentz algebra (the spinor representation).

Under a finite Lorentz transformation with parameters $\omega_{\mu\nu}$, spinors transform as follows:

$$\begin{aligned}\psi_\alpha &\mapsto \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha^{\ \beta}\psi_\beta && \text{(left-handed)} \\ \bar{\chi}^{\dot{\alpha}} &\mapsto \bar{\chi}^{\dot{\beta}}\ \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right)^{\dot{\alpha}}_{\ \dot{\beta}} && \text{(right-handed)}\end{aligned}$$

Now consider the spins with respect to the $SU(2)$s spanned by the A_i and B_i:

$$\begin{aligned}\psi_\alpha: &\qquad (A,\ B) = \left(\frac{1}{2},\ 0\right) &\Longrightarrow\quad J_i &= \frac{1}{2}\,\sigma_i\ , & K_i &= -\frac{i}{2}\,\sigma_i \\ \bar{\chi}^{\dot{\alpha}}: &\qquad (A,\ B) = \left(0,\ \frac{1}{2}\right) &\Longrightarrow\quad J_i &= \frac{1}{2}\,\sigma_i\ , & K_i &= +\frac{i}{2}\,\sigma_i\end{aligned}$$

Some useful identities concerning the σ^μ and $\sigma^{\mu\nu}$ can be found on the examples sheets. For now, let us just mention the identities[3]

$$\begin{aligned}\sigma^{\mu\nu} &= \frac{1}{2i}\,\epsilon^{\mu\nu\rho\sigma}\,\sigma_{\rho\sigma} \\ \bar{\sigma}^{\mu\nu} &= -\frac{1}{2i}\,\epsilon^{\mu\nu\rho\sigma}\,\bar{\sigma}_{\rho\sigma}\ ,\end{aligned}$$

known as *self duality* and *anti self duality*. They are important because naively $\sigma^{\mu\nu}$ being antisymmetric seems to have $\frac{4\times 3}{2}$ components, but the self duality conditions reduces this by half. A reference book illustrating many of the calculations for two - component spinors is [2].

2.1.4 Products of Weyl spinors

Define the product of two Weyl spinors as

$$\begin{aligned}\chi\psi &:= \chi^\alpha\,\psi_\alpha \;=\; -\chi_\alpha\,\psi^\alpha \\ \bar{\chi}\bar{\psi} &:= \bar{\chi}_{\dot{\alpha}}\,\bar{\psi}^{\dot{\alpha}} \;=\; -\bar{\chi}^{\dot{\alpha}}\,\bar{\psi}_{\dot{\alpha}}\ ,\end{aligned}$$

[3] $\epsilon_{0123} = 1 = -\epsilon^{0123}$

where in particular

$$\psi\psi \;=\; \psi^\alpha\,\psi_\alpha \;=\; \epsilon^{\alpha\beta}\,\psi_\beta\,\psi_\alpha \;=\; \psi_2\,\psi_1 \;-\; \psi_1\,\psi_2\ .$$

Choosing the ψ_α to be *anticommuting Grassmann numbers*, $\psi_1\psi_2 = -\psi_2\psi_1$, so $\psi\psi = 2\psi_2\psi_1$. Thus $\psi_\alpha\psi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}(\psi\psi)$.
We note that eq. 2.1 implies that $A \leftrightarrow B$ under Hermitian conjugation. Therefore, the Hermitian conjugate of a left (right)-handed spinor is a right (left)-handed spinor. Thus we define

$$(\psi_\alpha)^\dagger \;:=\; \bar{\psi}_{\dot\alpha}\ , \qquad \bar{\psi}^{\dot\alpha} \;:=\; \psi^*_\beta\,(\sigma^0)^{\beta\dot\alpha}$$

it follows that

$$(\chi\psi)^\dagger \;=\; \bar\chi\bar\psi\ , \qquad (\psi\,\sigma^\mu\,\bar\chi)^\dagger \;=\; \chi\,\sigma^\mu\,\bar\psi$$

which justifies the $\nearrow$ contraction of implicit dotted indices in contrast to the $\searrow$ implicit contraction of undotted ones.
In general we can generate all higher dimensional representations of the Lorentz group by products of the fundamental representation $(\frac{1}{2},\,0)$ and its conjugate $(0,\,\frac{1}{2})$. The computation of tensor products $(\frac{r}{2},\,\frac{s}{2}) = (\frac{1}{2},\,0)^{\otimes r}\otimes(0,\,\frac{1}{2})^{\otimes s}$ can be reduced to successive application of the elementary $SU(2)$ rule $(\frac{j}{2})\otimes(\frac{1}{2}) = (\frac{j-1}{2})\oplus(\frac{j+1}{2})$ (for $j \neq 0$).
Let us give two examples for tensoring Lorentz representations:

- $(\frac{1}{2},\,0)\otimes(0,\,\frac{1}{2}) \;=\; (\frac{1}{2},\,\frac{1}{2})$

  Bi-spinors with different chiralities can be expanded in terms of the $\sigma^\mu_{\alpha\dot\alpha}$. Actually, the σ^μ matrices form a complete orthonormal set of 2×2 matrices with respect to the trace $\mathrm{Tr}\{\sigma^\mu\bar\sigma^\nu\} = 2\eta^{\mu\nu}$:

  $$\psi_\alpha\,\bar\chi_{\dot\alpha} \;=\; \frac{1}{2}\,\left(\psi\,\sigma_\mu\,\bar\chi\right)\,\sigma^\mu_{\alpha\dot\alpha}$$

  Hence, two spinor degrees of freedom with opposite chirality give rise to a Lorentz vector $\psi\sigma_\mu\bar\chi$.

- $(\frac{1}{2},\,0)\otimes(\frac{1}{2},\,0) \;=\; (0,0)\oplus(1,0)$

  Alike bi-spinors require a different set of matrices to expand, $\epsilon_{\alpha\beta}$ and $(\sigma^{\mu\nu}\epsilon^T)_{\alpha\beta} := (\sigma^{\mu\nu})_\alpha{}^\gamma\epsilon_{\beta\gamma}$. The former represents the unique antisymmetric 2×2 matrix, the latter provides the symmetric ones.

  $$\psi_\alpha\,\chi_\beta \;=\; \frac{1}{2}\,\epsilon_{\alpha\beta}\,(\psi\chi) \;+\; \frac{1}{2}\,\left(\sigma^{\mu\nu}\,\epsilon^T\right)_{\alpha\beta}\,(\psi\,\sigma_{\mu\nu}\,\chi)$$

  The product of spinors with alike chiralities decomposes into two Lorentz irreducible representations, a scalar $\psi\chi$ and a self-dual antisymmetric rank two tensor $\psi\,\sigma_{\mu\nu}\,\chi$. The counting of independent components of $\sigma^{\mu\nu}$ from its self-duality property precisely provides the right number of three components for the $(1,0)$ representation. Similarly, there is an anti-self dual tensor $\bar\chi\bar\sigma^{\mu\nu}\bar\psi$ in $(0,1)$.

These expansions are also referred to as FIERZ identities.

2.1.5 Dirac spinors

To connect the ideas of Weyl spinors with the more standard Dirac theory, define

$$\gamma^\mu \;:=\; \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} ,$$

then these γ^μ satisfy the *Clifford algebra*

$$\left\{ \gamma^\mu \, , \, \gamma^\nu \right\} \;=\; 2\,\eta^{\mu\nu}\,\mathbb{1} \, .$$

The matrix γ^5, defined as

$$\gamma^5 \;:=\; i\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 \;=\; \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} ,$$

can have eigenvalues ± 1 (chirality). The generators of the Lorentz group are

$$\Sigma^{\mu\nu} \;=\; \frac{i}{4}\,\gamma^{\mu\nu} \;=\; \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix} .$$

We define *Dirac spinors* to be the direct sum of two Weyl spinors of opposite chirality,

$$\Psi_D \;:=\; \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} ,$$

such that the action of γ^5 is given as

$$\gamma^5\,\Psi_D \;=\; \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \;=\; \begin{pmatrix} -\psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} .$$

We can define the following projection operators P_L, P_R,

$$P_L \;:=\; \frac{1}{2}\left(\mathbb{1} \,-\, \gamma^5\right) , \qquad P_R \;:=\; \frac{1}{2}\left(\mathbb{1} \,+\, \gamma^5\right) ,$$

eliminating one part of definite chirality, i.e.

$$P_L\,\Psi_D \;=\; \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix} , \qquad P_R\,\Psi_D \;=\; \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} .$$

Finally, define the *Dirac conjugate* $\overline{\Psi}_D$ and *charge conjugate* spinor ${\Psi_D}^C$ by

$$\overline{\Psi}_D \;:=\; (\chi^\alpha, \, \bar{\psi}_{\dot{\alpha}}) \;=\; \Psi_D^\dagger\,\gamma^0$$

$${\Psi_D}^C \;:=\; C\,\overline{\Psi}_D^T \;=\; \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} ,$$

where C denotes the *charge conjugation matrix*

$$C \;:=\; \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} .$$

Majorana spinors Ψ_M have property $\psi_\alpha = \chi_\alpha$,

$$\Psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} = {\Psi_M}^C ,$$

so a general Dirac spinor (and its charge conjugate) can be decomposed as

$$\Psi_D = \Psi_{M1} + i\Psi_{M2} , \qquad {\Psi_D}^C = \Psi_{M1} - i\Psi_{M2} .$$

2.2 SUSY algebra

2.2.1 History of supersymmetry

- In the 1960's, from the study of strong interactions, many hadrons have been discovered and were successfully organised in multiplets of $SU(3)_f$, the f referring to flavour. This procedure was known as the *eightfold way* of Gell-Mann and Neeman. Questions arouse about bigger multiplets including particles of different spins.
- In a famous *No-go theorem* (Coleman, Mandula 1967) said that the most general symmetry of the S - matrix is Poincaré $\times$ internal, that cannot mix different spins (for example), if you still require there to be interactions
- Golfand and Licktman (1971) extended the Poincaré algebra to include spinor generators Q_α, where $\alpha = 1, 2$.
- Ramond,Neveu-Schwarz, Gervais, Sakita (1971): devised supersymmetry in 2 dimensions (from string theory).
- Wess and Zumino (1974) wrote down supersymmetric field theories in 4 dimensions. They opened the way for many other contributions to the field. This is often seen as the actual starting point on systematic study of supersymmetry.
- Haag, Lopuszanski, Sohnius (1975): generalised the Coleman Mandula theorem to show that the only non-trivial quantum field theories have a symmetry group of super Poincareé group in a direct product with internal symmetries.

2.2.2 Graded algebra

We wish to *extend* the Poincaré algebra non-trivially. The *Coleman Mandula theorem* stated that in 3+1 dimensions, one cannot do this in a non-trivial way and still have non-zero scattering amplitudes. In other words, there is no non-trivial mix of Poincaré and internal symmetries with non-zero scattering except for the direct product

Poincaré $\times$ internal.

However (as usual with no-go theorems) there was a loop-hole because of an implicit axiom: the proof only considered "*bosonic* generators".
We wish to turn bosons into fermions, thus we need to introduce a fermionic generator Q. Heuristically:

$$Q|\text{boson}\rangle \propto |\text{fermion}\rangle, \qquad Q|\text{fermion}\rangle \propto |\text{boson}\rangle.$$

For this, we require a graded algebra - a generalisation of Lie algebra. If O_a is an operator of an algebra (such as a group generator), a graded algebra is

$$O_a O_b - (-1)^{\eta_a \eta_b} O_b O_a = i C^e_{ab} O_e, \tag{2.3}$$

where $\eta_a = 0$ if O_a is a *bosonic generator*, and $\eta_a = 1$ if O_a is a *fermionic generator*.
For supersymmetry, the bosonic generators are the Poincaré generators P^μ, $M^{\mu\nu}$ and the fermionic generators Q^A_α, $\bar{Q}^A_{\dot{\alpha}}$, where $A = 1, ..., N$. In case $N = 1$ we speak of a simple SUSY, in case $N > 1$ of an extended SUSY. In this section, we will only discuss $N = 1$.

We know the commutation relations $[P^\mu, P^\nu]$, $[P^\mu, M^{\rho\sigma}]$ and $[M^{\mu\nu}, M^{\rho\sigma}]$ already from the Poincaré algebra, so we need to find

$$\text{(a)} \ \Big[Q_\alpha \ , \ M^{\mu\nu}\Big] \ , \ \text{(b)} \ \Big[Q_\alpha \ , \ P^\mu\Big] \ ,$$
$$\text{(c)} \ \Big\{Q_\alpha \ , \ Q_\beta\Big\} \ , \ \ \text{(d)} \ \Big\{Q_\alpha \ , \ \bar{Q}_{\dot{\beta}}\Big\} \ ,$$

also (for internal symmetry generators T_i)

$$\text{(e)} \ \Big[Q_\alpha \ , \ T_i\Big] \ .$$

We shall be using the fact that the right hand sides must be *linear* and that they must transform in the same way as the commutators under a Lorentz transformation, for instance. The relations for $Q \leftrightarrow \bar{Q}$ may then be obtained from these by taking hermitian conjugates.

- (a) $\Big[Q_\alpha \ , \ M^{\mu\nu}\Big]$: we can work this one out by knowing how Q_α transforms as a spinor and as an operator.

  Since Q_α is a spinor, it transforms under the exponential of the $SL(2, \mathbb{C})$ generators $\sigma^{\mu\nu}$:

  $$Q'_\alpha \ = \ \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha^{\ \beta} Q_\beta \ \approx \ \left(\mathbb{1} \ - \ \frac{i}{2} \ \omega_{\mu\nu} \, \sigma^{\mu\nu}\right)_\alpha^{\ \beta} Q_\beta \ .$$

  Under an active transformation, as an operator, $|\psi\rangle \to U|\psi\rangle \Rightarrow \langle\psi|Q_\alpha|\psi\rangle \to \langle\psi|U^\dagger Q_\alpha U|\psi\rangle$, where we set the right hand side equal to $\langle\psi|Q'_\alpha|\psi\rangle$, and where $U = \exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right)$. Hence

  $$Q'_\alpha \ = \ U^\dagger \, Q_\alpha \, U \ \approx \ \left(\mathbb{1} \ + \ \frac{i}{2} \ \omega_{\mu\nu} \, M^{\mu\nu}\right) Q_\alpha \left(\mathbb{1} \ - \ \frac{i}{2} \ \omega_{\mu\nu} \, M^{\mu\nu}\right) \ .$$

  Compare these two expressions for Q'_α up to first order in $\omega_{\mu\nu}$,

  $$Q_\alpha \ - \ \frac{i}{2} \ \omega_{\mu\nu} \ (\sigma^{\mu\nu})_\alpha^{\ \beta} \, Q_\beta \ = \ Q_\alpha \ - \ \frac{i}{2} \ \omega_{\mu\nu} \left(Q_\alpha \, M^{\mu\nu} \ - \ M^{\mu\nu} \, Q_\alpha\right) \ + \ \mathcal{O}(\omega^2)$$

  $$\Longrightarrow \ \boxed{\Big[Q_\alpha \ , \ M^{\mu\nu}\Big] \ = \ (\sigma^{\mu\nu})_\alpha^{\ \beta} \, Q_\beta}$$

  Similarly,

  $$\boxed{\Big[\bar{Q}^{\dot{\alpha}}, \ M^{\mu\nu}\Big] \ = \ (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} \, \bar{Q}^{\dot{\beta}}}$$

- (b) $\left[Q_\alpha\,,\,P^\mu\right]$: $c\cdot(\sigma^\mu)_{\alpha\dot\alpha}\bar Q^{\dot\alpha}$ is the only way of writing a sensible term with free indices μ, α which is linear in Q. To fix the constant c, consider $[\bar Q^{\dot\alpha}, P^\mu] = c^* \cdot (\bar\sigma^\mu)^{\dot\alpha\beta} Q_\beta$ (take adjoints using $(Q_\alpha)^\dagger = \bar Q_{\dot\alpha}$ and $(\sigma^\mu \bar Q)^\dagger_\alpha = (Q\sigma^\mu)_{\dot\alpha}$). The Jacobi identity for P^μ, P^ν and Q_α

$$\begin{aligned} 0 \;&=\; \left[P^\mu\,,\,\left[P^\nu\,,\,Q_\alpha\right]\right] \;+\; \left[P^\nu\,,\,\left[Q_\alpha\,,\,P^\mu\right]\right] \;+\; \left[Q_\alpha\,,\,\underbrace{\left[P^\mu\,,\,P^\nu\right]}_{0}\right] \\ &=\; -c\,(\sigma^\nu)_{\alpha\dot\alpha}\left[P^\mu\,,\,\bar Q^{\dot\alpha}\right] \;+\; c\,(\sigma^\mu)_{\alpha\dot\alpha}\left[P^\nu\,,\,\bar Q^{\dot\alpha}\right] \\ &=\; |c|^2\,(\sigma^\nu)_{\alpha\dot\alpha}\,(\bar\sigma^\mu)^{\dot\alpha\beta}\,Q_\beta \;-\; |c|^2\,(\sigma^\mu)_{\alpha\dot\alpha}\,(\bar\sigma^\nu)^{\dot\alpha\beta}\,Q_\beta \\ &=\; |c|^2\,\underbrace{(\sigma^\nu\,\bar\sigma^\mu \;-\; \sigma^\mu\,\bar\sigma^\nu)_\alpha{}^\beta}_{\neq 0}\,Q_\beta \end{aligned}$$

  can only hold for general Q_β, if $c = 0$, so

$$\boxed{\left[Q_\alpha\,,\,P^\mu\right] \;=\; \left[\bar Q^{\dot\alpha}\,,\,P^\mu\right] \;=\; 0}$$

- (c) $\left\{Q_\alpha\,,\,Q_\beta\right\}$
  Due to index structure, that commutator should look like

$$\left\{Q_\alpha\,,\,Q^\beta\right\} \;=\; k\,(\sigma^{\mu\nu})_\alpha{}^\beta\,M_{\mu\nu}\;.$$

  Since the left hand side commutes with P^μ and the right hand side doesn't, the only consistent choice is $k = 0$, i.e.

$$\boxed{\left\{Q_\alpha\,,\,Q_\beta\right\} \;=\; 0, \qquad \left\{\bar Q_{\dot\alpha}\,,\,\bar Q_{\dot\beta}\right\} \;=\; 0}$$

- (d) $\left\{Q_\alpha\,,\,\bar Q_{\dot\beta}\right\}$
  This time, index structure implies an ansatz

$$\left\{Q_\alpha\,,\,\bar Q_{\dot\beta}\right\} \;=\; t\,(\sigma^\mu)_{\alpha\dot\beta}\,P_\mu\;.$$

  There is no way of fixing t, so, by convention, set $t = 2$, defining the normalisation of the operators:

$$\boxed{\left\{Q_\alpha\,,\,\bar Q_{\dot\beta}\right\} \;=\; 2\,(\sigma^\mu)_{\alpha\dot\beta}\,P_\mu}$$

Notice that two symmetry transformations $Q_\alpha \bar Q_{\dot\beta}$ have the effect of a translation. Let $|B\rangle$ be a bosonic state and $|F\rangle$ a fermionic one, then

$$Q_\alpha\,|F\rangle \;=\; |B\rangle\;, \qquad \bar Q_{\dot\beta}\,|B\rangle \;=\; |F\rangle \quad\Longrightarrow\quad Q\bar Q:\; |B\rangle \;\mapsto\; |B\text{ (translated)}\rangle\;.$$

- (e) $\Big[Q_\alpha \ , \ T_i\Big]$

  Usually, this commutator vanishes due to the Coleman-Mandula theorem. Exceptions are $U(1)$ automorphisms of the supersymmetry algebra known as R *symmetry*. The algebra is invariant under the simultaneous change

$$Q_\alpha \ \mapsto \ \exp(i\lambda)\, Q_\alpha \ , \qquad \bar{Q}_{\dot{\alpha}} \ \mapsto \ \exp(-i\lambda)\, \bar{Q}_{\dot{\alpha}} \ .$$

  Let R be a global $U(1)$ generator, then, since $Q_\alpha \mapsto e^{-iR\lambda} Q_\alpha e^{iR\lambda}$,

$$\boxed{\Rightarrow \Big[Q_\alpha \ , \ R\Big] \ = \ Q_\alpha \ , \qquad \Big[\bar{Q}_{\dot{\alpha}} \ , \ R\Big] \ = \ -\bar{Q}_{\dot{\alpha}}.}$$

2.3 Representations of the Poincaré group

Since we are changing the Poincaré group, we must check to see if anything happens to the Casimirs of the changed group, since these are used to label irreducible representations (remember that one needs a complete commuting set of observables to label them). Recall the rotation group $\{J_i : \ i = 1, 2, 3\}$ satisfying

$$\Big[J_i \ , \ J_j\Big] \ = \ i\epsilon_{ijk}\, J_k \ .$$

The Casimir operator

$$J^2 \ = \ \sum_{i=1}^{3} J_i^2$$

commutes with all the J_i and labels irreducible representations by eigenvalues $j(j+1)$ of J^2. Within these irreducible representations, the J_3 eigenvalues $j_3 = -j, -j+1, ..., j-1, j$ label each element. States are labelled like $|j, j_3\rangle$.
Also recall the two Casimirs in the Poincaré group, one of which involves the *Pauli Ljubanski vector* W_μ describing generalised spin

$$W_\mu \ = \ \frac{1}{2}\, \epsilon_{\mu\nu\rho\sigma}\, P^\nu\, M^{\rho\sigma}$$

(where $\epsilon_{0123} = -\epsilon^{0123} = +1$).

The Poincaré Casimirs are then given by

$$C_1 \ = \ P^\mu\, P_\mu \ , \qquad C_2 \ = \ W^\mu\, W_\mu,$$

since the C_i commute with all generators.

Poincaré multiplets are labelled $|m, \omega\rangle$, where m^2 is the eigenvalue of C_1 and ω is the eigenvalue of C_2. States within those irreducible representations carry the eigenvalue p^μ of the generator P^μ as a label. Notice that at this level the Pauli Ljubanski vector only provides a short way to express the second Casimir. Even though W_μ has standard commutation relations with the generators of the Poincaré group $M_{\mu\nu}$ (since it transforms as a vector under Lorentz transformations) and commutes with P_μ (it is invariant under translations),

the commutator $[W_\mu, W_\nu] = i\epsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma$ implies that the W_μ's by themselves are not generators of a closed algebra.
To find more labels we take P^μ as given and look for all elements of the Lorentz group that commute with P^μ. This defines little groups:

- Massive particles, $p^\mu = (m, \underbrace{0,\ 0,\ 0}_{\text{invariant under rot.}})$, have rotations as their little group, since they leave p^μ invariant. From the definition of W_μ, it follows that

$$W_0 \ = \ 0 \ , \qquad W_i \ = \ -m\, J_i \ .$$

  Thus, $C_1 = P^2$ with eigenvalue m^2, $C_2 = -P^2 J^2$ with eigenvalue $-m^2 j(j+1)$, hence a particle with non-zero mass is an irreducible representation of the Poincaré group with labels $|m, j; p^\mu, j_3\rangle$.

- Massless particles have $p^\mu = (|\mathbf{p}|, \ \mathbf{p})$ and W^μ eigenvalues λp^μ (see the Part III Particles and Symmetries course). Thus, $\lambda = \mathbf{j} \cdot \mathbf{p}/|\mathbf{p}|$ is the helicity.

  States are thus labelled $|0, 0; p^\mu, \lambda\rangle =: |p^\mu, \lambda\rangle$. Under CPT[4], those states transform to $|p^\mu, -\lambda\rangle$. λ must be integer or half integer[5] $\lambda = 0, \frac{1}{2}, 1, ...,$ e.g. $\lambda = 0$ (Higgs), $\lambda = \frac{1}{2}$ (quarks, leptons), $\lambda = 1$ (γ, $W^\pm$, Z^0, g) and $\lambda = 2$ (graviton). Note that massive representations are CPT self-conjugate.

2.4 $\mathcal{N} = 1$ supersymmetry representations

For $\mathcal{N} = 1$ supersymmetry, $C_1 = P^\mu P_\mu$ is still a good Casimir, $C_2 = W^\mu W_\mu$, however, is not. One can have particles of different spin within one multiplet. To get a new Casimir $\tilde{C}_2$ (corresponding to superspin), we define

$$B_\mu \ := \ W_\mu \ - \ \frac{1}{4}\, \bar{Q}_{\dot{\alpha}}\, (\bar{\sigma}_\mu)^{\dot{\alpha}\beta}\, Q_\beta \ , \qquad C_{\mu\nu} \ := \ B_\mu\, P_\nu \ - \ B_\nu\, P_\mu$$

$$\tilde{C}_2 \ := \ C_{\mu\nu}\, C^{\mu\nu} \ .$$

2.4.1 Bosons and fermions in a supermultiplet

In any supersymmetric multiplet, the number n_B of bosons equals the number n_F of fermions,

$$n_B \ = \ n_F \ .$$

To prove this, consider the *fermion number operator* $(-1)^F = (-)^F$, defined via

$$(-)^F\, |B\rangle \ = \ |B\rangle \ , \qquad (-)^F\, |F\rangle \ = \ -|F\rangle \ .$$

This new operator $(-)^F$ anticommutes with Q_α since

$$(-)^F\, Q_\alpha\, |F\rangle \ = \ (-)^F\, |B\rangle \ = \ |B\rangle \ = \ Q_\alpha\, |F\rangle \ = \ -Q_\alpha\, (-)^F\, |F\rangle \ \Longrightarrow \ \Big\{ (-)^F \, , \, Q_\alpha \Big\} \ = \ 0 \, .$$

[4]See the Standard Model Part III course for a rough proof of the CPT theorem, which states that *any local Lorentz invariant quantum field theory is CPT invariant.*

[5]See the Part II Principles of Quantum Mechanics course.

Next, consider the trace (in the operator sense, i.e. over elements of the multiplet)

$$\begin{aligned}\mathrm{Tr}\left\{(-)^F\left\{Q_\alpha\,,\;\bar{Q}_{\dot\beta}\right\}\right\} \;&=\; \mathrm{Tr}\Big\{\underbrace{(-)^F\,Q_\alpha}_{\text{anticommute}}\;\bar{Q}_{\dot\beta}\;+\;\underbrace{(-)^F\,\bar{Q}_{\dot\beta}\,Q_\alpha}_{\text{cyclic perm.}}\Big\}\\ &=\; \mathrm{Tr}\Big\{-Q_\alpha\,(-)^F\,\bar{Q}_{\dot\beta}\;+\;Q_\alpha\,(-)^F\,\bar{Q}_{\dot\beta}\Big\}\;=\;0\;.\end{aligned}$$

On the other hand, it can be evaluated using $\{Q_\alpha,\bar{Q}_{\dot\beta}\}=2(\sigma^\mu)_{\alpha\dot\beta}P_\mu$,

$$\mathrm{Tr}\left\{(-)^F\left\{Q_\alpha\,,\;\bar{Q}_{\dot\beta}\right\}\right\}\;=\;\mathrm{Tr}\left\{(-)^F\,2\,(\sigma^\mu)_{\alpha\dot\beta}\,P_\mu\right\}\;=\;2\,(\sigma^\mu)_{\alpha\dot\beta}\,p_\mu\,\mathrm{Tr}\Big\{(-)^F\Big\}\;,$$

where P^μ is replaced by its eigenvalues p^μ for the specific state. The conclusion is

$$\begin{aligned}0\;=\;\mathrm{Tr}\Big\{(-)^F\Big\}\;&=\;\sum_{\text{bosons}}\langle B|\,(-)^F\,|B\rangle\;+\;\sum_{\text{fermions}}\langle F|\,(-)^F\,|F\rangle\\ &=\;\sum_{\text{bosons}}\langle B|B\rangle\;-\;\sum_{\text{fermions}}\langle F|F\rangle\;=\;n_B\;-\;n_F\;.\end{aligned}$$

$\mathrm{Tr}\Big\{(-)^F\Big\}$ is known as the "Witten index".

2.4.2 Massless supermultiplet

States of massless particles have P_μ - eigenvalues $p_\mu\;=\;(E,\;0,\;0,\;E)$. The Casimirs $C_1=P^\mu P_\mu$ and $\tilde{C}_2=C_{\mu\nu}C^{\mu\nu}$ are zero. Consider the algebra (implicitly acting on our massless state $|p^\mu,\;\lambda\rangle$ on the right hand side)

$$\left\{Q_\alpha\,,\;\bar{Q}_{\dot\beta}\right\}\;=\;2\,(\sigma^\mu)_{\alpha\dot\beta}\,P_\mu\;=\;2\,E\,\left(\sigma^0\,+\,\sigma^3\right)_{\alpha\dot\beta}\;=\;4\,E\begin{pmatrix}1&0\\0&0\end{pmatrix}_{\alpha\dot\beta}\;,$$

which implies that Q_2 is zero in the representation:

$$\langle p^\mu,\lambda|\left\{Q_2\,,\;\bar{Q}_{\dot 2}\right\}|p^\mu,\;\lambda\rangle\;=\;0\Leftrightarrow\;\bar{Q}_{\dot 2}|p^\mu,\;\lambda\rangle\;=\;Q_2|p^\mu,\;\lambda\rangle\;=\;0.$$

We may also find one element $|p^\mu,\lambda\rangle$ such that $Q_1|p^\mu,\lambda\rangle=0$.
From our previous commutation relation,

$$[W_\mu,\;\bar{Q}^{\dot\alpha}]=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu[M^{\rho\sigma},\bar{Q}^{\dot\alpha}]=-\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu(\bar\sigma^{\rho\sigma})^{\dot\alpha}{}_{\dot\beta}\bar{Q}^{\dot\beta}\tag{2.4}$$

and the definition of W_μ, in this representation

$$\Rightarrow[W_0,\;\bar{Q}^{\dot\alpha}]|p^\mu,\;\lambda\rangle=-\frac{i}{8}\epsilon_{03jk}p^3\left([\bar\sigma^j,\;\sigma^k]\bar{Q}\right)^{\dot\alpha}|p^\mu,\;\lambda\rangle=-\frac{1}{2}p^3(\sigma^3\bar{Q})^{\dot\alpha}|p^\mu,\;\lambda\rangle.\tag{2.5}$$

So, remembering that $p^3=-p_0$ and, for massless representations, $W_0|p^\mu,\;\lambda\rangle=\lambda p_0|p^\mu,\lambda\rangle$,

$$W_0\bar{Q}^{\dot 2}|p^\mu,\;\lambda\rangle=\left([W_0,\bar{Q}^{\dot 2}]+\bar{Q}^{\dot 2}\lambda p_0\right)|p^\mu,\;\lambda\rangle=(\lambda-\frac{1}{2})p_0\bar{Q}^{\dot 2}|p^\mu,\;\lambda\rangle.$$

Thus, $\bar{Q}^{\dot{2}} = -\bar{Q}_{\dot{1}}$ decreases the helicity by 1/2 a unit[6]. The normalised state is then

$$|p^\mu,\ \lambda - \frac{1}{2}\rangle = \frac{\bar{Q}_{\dot{1}}}{\sqrt{4E}}|p^\mu,\ \lambda\rangle \tag{2.6}$$

and there are no other states, since Eq. 2.6 $\Rightarrow \bar{Q}_{\dot{1}}|p^\mu,\ \lambda - \frac{1}{2}\rangle = 0$ and

$$Q_1|p^\mu,\ \lambda - \frac{1}{2}\rangle = \frac{1}{\sqrt{4E}}Q_1\bar{Q}_{\dot{1}}|p^\mu,\ \lambda\rangle = \frac{1}{\sqrt{4E}}\left(\{Q_1,\ \bar{Q}_{\dot{1}}\} - \bar{Q}_{\dot{1}}Q_1\right)|p^\mu,\ \lambda\rangle = \sqrt{4E}|p^\mu,\ \lambda\rangle,$$

Thus, we have two states in the supermultiplet: a boson and a fermion, plus CPT conjugates:

$$|p^\mu, \pm\lambda\rangle\ , \qquad |p^\mu, \pm\left(\lambda - \tfrac{1}{2}\right)\rangle\ .$$

There are, for example, chiral multiplets with $\lambda = 0, \frac{1}{2}$, vector- or gauge multiplets ($\lambda = \frac{1}{2}, 1$ gauge and gaugino)

$\lambda = 0$ scalar	$\lambda = \frac{1}{2}$ fermion
squark	quark
slepton	lepton
Higgs	Higgsino

$\lambda = \frac{1}{2}$ fermion	$\lambda = 1$ boson
photino	photon
gluino	gluon
Wino, Zino	$W,\ Z$

,

as well as the graviton with its partner:

$\lambda = \frac{3}{2}$ fermion	$\lambda = 2$ boson
gravitino	graviton

Question: Why do we put matter fields in the $\lambda = \{0, \frac{1}{2}\}$ supermultiplets rather than in the $\lambda = \{\frac{1}{2},\ 1\}$ ones?

2.4.3 Massive supermultiplet

In case of $m \neq 0$, in the centre of mass frame there are P^μ - eigenvalues $p^\mu = (m,\ 0,\ 0,\ 0)$ and Casimirs

$$C_1 \ = \ P^\mu P_\mu \ = \ m^2\ , \qquad \tilde{C}_2 \ = \ C_{\mu\nu}C^{\mu\nu} \ = \ 2m^4\,Y^i\,Y_i\ ,$$

where Y_i denotes superspin

$$Y_i \ = \ J_i \ - \ \frac{1}{4m}\bar{Q}\,\bar{\sigma}_i\,Q\ , \qquad \left[Y_i\ ,\ Y_j\right] \ = \ i\epsilon_{ijk}\,Y_k\ .$$

The eigenvalues of $Y^2 = Y^iY_i$ are $y(y+1)$, so we label irreducible representations by $|m, y\rangle$. Again, the anticommutation - relation for Q and $\bar{Q}$ is the key to get the states:

$$\left\{Q_\alpha\ ,\ \bar{Q}_{\dot{\beta}}\right\} \ = \ 2\,(\sigma^\mu)_{\alpha\dot{\beta}}\,P_\mu \ = \ 2\,m\,(\sigma^0)_{\alpha\dot{\beta}} \ = \ 2\,m\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}_{\alpha\dot{\beta}}$$

[6]Note that we have used natural units, therefore $\hbar = 1$.

Let $|\Omega\rangle$ be the ground state, annihilated by $Q_{1,2}$. Consequently,

$$Y_i\,|\Omega\rangle \;=\; J_i\,|\Omega\rangle \;-\; \frac{1}{4m}\,\bar{Q}\,\bar{\sigma}_i\,\underbrace{Q|\Omega\rangle}_{0} \;=\; J_i\,|\Omega\rangle\;,$$

i.e. for $|\Omega\rangle$, the spin j and superspin y are the same. So for given m, y:

$$|\Omega\rangle \;=\; |m, j = y; p^\mu, j_3\rangle$$

We may obtain the rest of the supersymmetry multiplet by deriving the commutation relations

$$[Q_\alpha,\ J_i] = \frac{1}{2}(\sigma_i)_\alpha^{\beta} Q_\beta, \qquad [J_i,\ \bar{Q}^{\dot\alpha}] = -\frac{1}{2}(\sigma_i)^{\dot\alpha}_{\dot\beta}\bar{Q}^{\dot\beta} \tag{2.7}$$

from the supersymmetry algebra. Thus,

$$a_1^\dagger|j_3\rangle := \frac{\bar{Q}^{\dot 1}}{\sqrt{2m}}|j_3\rangle = |j_3 - \frac{1}{2}\rangle, \qquad a_2^\dagger|j_3\rangle := \frac{\bar{Q}^{\dot 2}}{\sqrt{2m}}|j_3\rangle = |j_3 + \frac{1}{2}\rangle. \tag{2.8}$$

We may use Eq. 2.7 to derive

$$[J^2, \bar{Q}^{\dot\alpha}] = \frac{3}{4}\bar{Q}^{\dot\alpha} - (\sigma_i)^{\dot\alpha}_{\dot\beta}\bar{Q}^{\dot\beta} J_i, \tag{2.9}$$

$$[J_3, a_1^\dagger a_2^\dagger] = [J^2, a_1^\dagger a_2^\dagger] = 0$$

(a) $\mathbf{y} = 0$

Let us now consider a specific case, $y = 0$. We define $J_\pm := J_1 \pm iJ_2$, which lowers/raises spin by 1 unit in the third direction (see Part II Principles of Quantum Mechanics notes) but leaves the total spin unchanged. Using Eq. 2.9, and $|\Omega\rangle := |m, 0, 0\rangle$,

$$J^2 a_1^\dagger|\Omega\rangle = \frac{3}{4}a_1^\dagger|\Omega\rangle - a_2^\dagger\underbrace{J_-|\Omega\rangle}_{zero} - a_1^\dagger\underbrace{J_3|\Omega\rangle}_{zero} =: j(j+1)\bar{a}_1^\dagger|\Omega\rangle.$$

Hence $a_1^\dagger|\Omega\rangle$ has $j = 1/2$ and you can check that $j_3 = -1/2$. Similarly, $a_2^\dagger|\Omega\rangle = |m, 1/2, 1/2\rangle$. The remaining state

$$|\Omega'\rangle := a_2^\dagger\,a_1^\dagger\,|\Omega\rangle \;=\; -a_1^\dagger\,a_2^\dagger\,|\Omega\rangle$$

represents a different spin j object.

Question: How do we know that $|\Omega'\rangle \neq |\Omega\rangle$?

Thus, for the case $y = 0$, we have states

$$\begin{aligned} |\Omega\rangle \;&=\; |m, j = 0; p^\mu, j_3 = 0\rangle \\ a_{1,2}^\dagger\,|\Omega\rangle \;&=\; |m, j = \tfrac{1}{2}; p^\mu, j_3 = \pm\tfrac{1}{2}\rangle \\ a_2^\dagger\,a_1^\dagger\,|\Omega\rangle \;&=\; |m, j = 0; p^\mu, j_3 = 0\rangle \;\;=:\;\; |\Omega'\rangle \end{aligned}$$

(b) $\mathbf{y} \neq 0$

The case $y \neq 0$ proceeds slightly differently. The doublet $\bar{Q}_{\dot{\alpha}}$ is a doublet (i.e. spin 1/2 representation) of the right-handed $SU(2)$ in $SL(2,\mathbb{C})$, as Eq. 2.1.2 shows. The doublet $(a_1^\dagger,\ a_2^\dagger)$ acting on $|\Omega\rangle$ behaves like the combination of two spins: $\frac{1}{2}$ and j, from Eq. 2.8. This yields a linear combination of two possible total spins $j+\frac{1}{2}$ and $j-\frac{1}{2}$ with Clebsch Gordan coefficients k_i (recall $j \otimes \frac{1}{2} = (j-\frac{1}{2}) \oplus (j+\frac{1}{2})$):

$$\begin{aligned} a_2^\dagger\,|\Omega\rangle &= k_1\,|m, j = y+\tfrac{1}{2}; p^\mu, j_3+\tfrac{1}{2}\rangle \ + \ k_2\,|m, j = y-\tfrac{1}{2}; p^\mu, j_3+\tfrac{1}{2}\rangle \\ a_1^\dagger\,|\Omega\rangle &= k_3\,|m, j = y+\tfrac{1}{2}; p^\mu, j_3-\tfrac{1}{2}\rangle \ + \ k_4\,|m, j = y-\tfrac{1}{2}; p^\mu, j_3-\tfrac{1}{2}\rangle \ . \end{aligned}$$

We also have $a_1^\dagger|j_3\rangle = |j_3-\frac{1}{2}\rangle$ and $a_2^\dagger|j_3\rangle = |j_3+\frac{1}{2}\rangle$. In total, we have

$$\underbrace{2\cdot|m, j = y; p^\mu, j_3\rangle}_{(4y+2)\ \text{states}}\ , \qquad \underbrace{1\cdot|m, j = y+\tfrac{1}{2}; p^\mu, j_3\rangle}_{(2y+2)\ \text{states}}\ , \qquad \underbrace{1\cdot|m, j = y-\tfrac{1}{2}; p^\mu, j_3\rangle}_{(2y)\ \text{states}}\ ,$$

in a $|m, y\rangle$ multiplet, which is of course an equal number of bosonic and fermionic states. Notice that in labelling the states we have the value of m and y **fixed** throughout the multiplet and the values of j change state by state (as is proper, since in a supersymmetric multiplet there are states of different spin).

2.4.4 Parity

Parity interchanges $(A,\ B) \leftrightarrow (B,\ A)$, i.e. $(\frac{1}{2},\ 0) \leftrightarrow (0,\ \frac{1}{2})$. Since $\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu$, we need the following transformation rules for Q_α and $\bar{Q}_{\dot{\alpha}}$ under parity $\hat{P}$ (with phase factor η_P such that $|\eta_P| = 1$):

$$\begin{aligned} \hat{P}\,Q_\alpha\,\hat{P}^{-1} &= \eta_P\,(\sigma^0)_{\alpha\dot{\beta}}\,\bar{Q}^{\dot{\beta}} \\ \hat{P}\,\bar{Q}^{\dot{\alpha}}\,\hat{P}^{-1} &= -\eta_P^*\,(\bar{\sigma}^0)^{\dot{\alpha}\beta}\,Q_\beta \end{aligned}$$

This ensures $\hat{P}\ P^\mu\ \hat{P}^{-1} = (P^0\ ,\ -\vec{P})$ (see question on Examples Sheet I). and has the effect that $\hat{P}^2 Q_\alpha \hat{P}^{-2} = -Q_\alpha$. Moreover, consider the two $j=0$ massive states $|\Omega\rangle$ and $|\Omega'\rangle$: Since $\bar{Q}_{\dot{\alpha}}|\Omega'\rangle = 0$, whereas $Q_\alpha|\Omega\rangle = 0$, and since parity swaps $Q_\alpha \leftrightarrow \bar{Q}_{\dot{\alpha}}$, it also swaps $|\Omega\rangle \leftrightarrow |\Omega'\rangle$. To get ground states with a defined parity, we need linear combinations

$$|\pm\rangle \ := \ |\Omega\rangle \ \pm \ |\Omega'\rangle\ , \qquad \hat{P}\,|\pm\rangle \ = \ \pm\,|\pm\rangle\ .$$

These states are called scalar ($|+\rangle$) and pseudo-scalar ($|-\rangle$) states.

2.5 Extended supersymmetry

Having discussed the algebra and representations of simple ($N=1$) supersymmetry, we will turn now to the more general case of *extended supersymmetry* $N > 1$.

2.5.1 Algebra of extended supersymmetry

Now, the spinor generators get an additional label $A, B = 1, 2, ..., N$. The algebra is the same as for $N = 1$ except for

$$\boxed{\begin{aligned} \left\{ Q_\alpha^A \, , \, \bar{Q}_{\dot\beta B} \right\} \ &= \ 2 \, (\sigma^\mu)_{\alpha\dot\beta} \, P_\mu \, \delta^A{}_B \\ \left\{ Q_\alpha^A \, , \, Q_\beta^B \right\} \ = \ \epsilon_{\alpha\beta} \, Z^{AB}, \ \left\{ \bar{Q}_{\dot\alpha}^A \, , \, \bar{Q}_{\dot\beta}^B \right\} \ &= \ \epsilon_{\dot\alpha\dot\beta} \, (Z^\dagger)^{AB} \end{aligned}}$$

with antisymmetric *central charges* $Z^{AB} = -Z^{BA}$ commuting with all the generators

$$\left[Z^{AB} \, , \, P^\mu \right] \ = \ \left[Z^{AB} \, , \, M^{\mu\nu} \right] \ = \ \left[Z^{AB} \, , \, Q_\alpha^A \right] \ = \ \left[Z^{AB} \, , \, Z^{CD} \right] \ = \ \left[Z^{AB} \, , \, T_a \right] \ = \ 0 \, .$$

They form an abelian invariant sub-algebra of internal symmetries. Recall that $[T_a, T_b] = iC_{abc}T_c$. Let G be an internal symmetry group, then define the *R symmetry* $H \subset G$ to be the set of G elements that do not commute with the supersymmetry generators, e.g. $T_a \in G$ satisfying

$$\left[Q_\alpha^A \, , \, T_a \right] \ = \ S_a{}^A{}_B \, Q_\alpha^B \ \neq \ 0$$

is an element of H. If the eigenvalues of Z^{AB} are all zero, then the R symmetry is $H = U(N)$, but with some eigenvalues of $Z^{AB} \neq 0$, H will be a subgroup of $U(N)$. The existence of central charges is the main new ingredient of extended supersymmetries. The derivation of the previous algebra is a straightforward generalisation of the one for $N = 1$ supersymmetry.

2.5.2 Massless representations of $\mathcal{N} > 1$ supersymmetry

As we did for $N = 1$, we will proceed now to discuss massless and massive representations. We will start with the massless case which is simpler and has very important implications. Let $p_\mu = (E, \ 0, \ 0, \ E)$, then (similar to $N = 1$).

$$\left\{ Q_\alpha^A \, , \, \bar{Q}_{\dot\beta B} \right\} |p^\mu, \ \lambda\rangle \ = \ 4 \, E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta} \delta_B^A |p^\mu, \ \lambda\rangle \quad \Longrightarrow \quad Q_2^A |p^\mu, \ \lambda\rangle \ = \ 0$$

We can immediately see from this that the central charges Z^{AB} vanish since $Q_2^A |p^\mu, \ \lambda\rangle = 0$ implies $Z^{AB}|p^\mu, \ \lambda\rangle = 0$ from the anticommutator $\left\{ Q_1^A \, , \, Q_2^B \right\} |p^\mu, \ \lambda\rangle = 0 = \epsilon_{12} Z^{AB} |p^\mu, \ \lambda\rangle$. In order to obtain the full representation, we now define N creation- and N annihilation - operators

$$a^{A\dagger} \ := \ \frac{Q_1^A}{2\sqrt{E}} \, , \qquad a^A \ := \ \frac{\bar{Q}_{\dot 1}^A}{2\sqrt{E}} \quad \Longrightarrow \quad \left\{ a^A \, , \, a_B^\dagger \right\} \ = \ \delta^A{}_B \, ,$$

to get the following states (starting from ground state $|\Omega\rangle$, which is annihilated by all the a^A):

states	helicity	number of states
$\|\Omega\rangle$	λ_0	$1 = \binom{N}{0}$
$a^{A\dagger}\|\Omega\rangle$	$\lambda_0 + \frac{1}{2}$	$N = \binom{N}{1}$
$a^{A\dagger}a^{B\dagger}\|\Omega\rangle$	$\lambda_0 + 1$	$\frac{1}{2!}N(N-1) = \binom{N}{2}$
$a^{A\dagger}a^{B\dagger}a^{C\dagger}\|\Omega\rangle$	$\lambda_0 + \frac{3}{2}$	$\frac{1}{3!}N(N-1)(N-2) = \binom{N}{3}$
$\vdots$	$\vdots$	$\vdots$
$a^{N\dagger}a^{(N-1)\dagger}...a^{1\dagger}\|\Omega\rangle$	$\lambda_0 + \frac{N}{2}$	$1 = \binom{N}{N}$

Note that the total number of states is given by

$$\sum_{k=0}^{N}\begin{pmatrix} N \\ k \end{pmatrix} = \sum_{k=0}^{N}\begin{pmatrix} N \\ k \end{pmatrix} 1^k\, 1^{N-k} = 2^N .$$

Consider the following examples

- $N = 2$ vector multiplet, as shown in Fig. 2a: so-called because it contains a vector particle, which must be in the adjoint (i.e. vector-like, or real) representation if the quantum field theory is to be renormalisable. We can see that this $N = 2$ multiplet can be decomposed in terms of $N = 1$ multiplets: one $N = 1$ vector and one $N = 1$ chiral multiplet.

- $N = 2$ CPT self-conjugate hyper - multiplet, see Fig. 2b. Again this can be decomposed in terms of two $N = 1$ multiplets: one chiral, one anti-chiral.

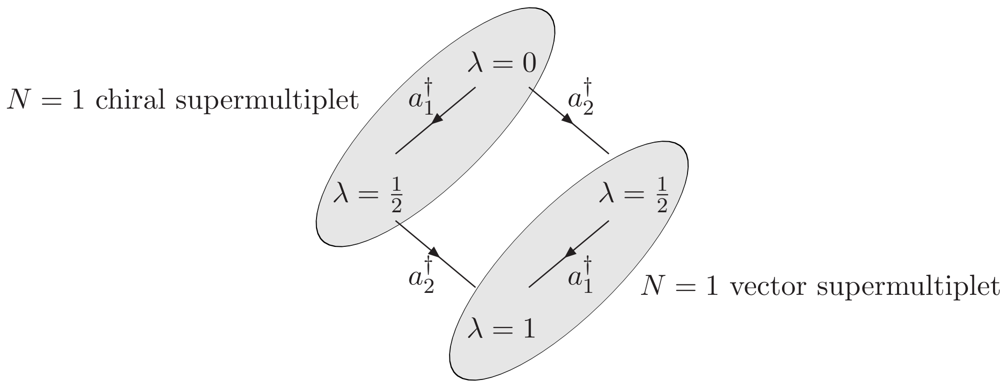
- $N = 4$ vector - multiplet ($\lambda_0 = -1$)

$$\begin{array}{rl} 1\times & \lambda = -1 \\ 4\times & \lambda = -\frac{1}{2} \\ 6\times & \lambda = \pm 0 \\ 4\times & \lambda = +\frac{1}{2} \\ 1\times & \lambda = +1 \end{array}$$

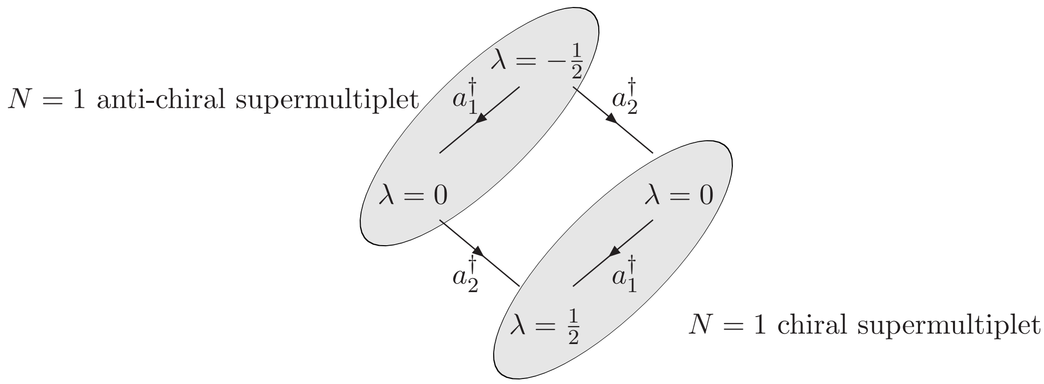
  This is the single $N = 4$ multiplet with states with $|\lambda| < \frac{3}{2}$. It consists of one $N = 2$ vector supermultiplet plus a CPT conjugate and two $N = 2$ hypermultiplets. Equivalently, it consists of one $N = 1$ vector and three $N = 1$ chiral supermultiplets plus their CPT conjugates.

- $N = 8$ maximum - multiplet ($\lambda_0 = -2$)

$$\begin{array}{rl} 1\times & \lambda = \pm 2 \\ 8\times & \lambda = \pm\frac{3}{2} \\ 28\times & \lambda = \pm 1 \\ 56\times & \lambda = \pm\frac{1}{2} \\ 70\times & \lambda = \pm 0 \end{array}$$



(a) Vector supermultiplet



(b) hyper supermultiplet

Figure 2. $N=2$ vector and hyper multiplets.

From these results we can extract very important general conclusions:

- In every multiplet: $\lambda_{\text{max}} - \lambda_{\text{min}} \quad = \quad \frac{N}{2}$
- Renormalisable theories have $|\lambda| \leq 1$ implying $N \leq 4$. Therefore $N = 4$ supersymmetry is the largest supersymmetry for renormalisable field theories. Gravity is not renormalisable!
- *The maximum number of supersymmetries is* $N = 8$. There is a strong belief that no massless particles of helicity $|\lambda| > 2$ exist (so only have $N \leq 8$). One argument against $|\lambda| > 2$ is the fact that massless particles of $|\lambda| > \frac{1}{2}$ and low momentum couple to some conserved currents ($\partial_\mu j^\mu = 0$ in $\lambda = \pm 1$ - electromagnetism, $\partial_\mu T^{\mu\nu}$ in $\lambda = \pm 2$ - gravity). But there are no conserved currents for $|\lambda| > 2$ (something that can also be seen from the Coleman Mandula theorem). Also, $N > 8$ would imply that there is more than one graviton. See chapter 13 in [4] on soft photons for a detailed discussion of this and the extension of his argument to supersymmetry in an

article by Grisaru and Pendleton (1977). Notice this is not a full no-go theorem, in particular the limit of low momentum had to be assumed.

- $N > 1$ *supersymmetries are non-chiral.* We know that the Standard Model particles live on complex fundamental representations. They are chiral since right handed quarks and leptons do not feel the weak interactions whereas left-handed ones do feel it (they are doublets under $SU(2)_L$). All $N > 1$ multiplets, except for the $N = 2$ hypermultiplet, have $\lambda = \pm 1$ particles transforming in the adjoint representation which is non-chiral. Then the $\lambda = \pm\frac{1}{2}$ particles within the multiplet would transform in the same representation and therefore be non-chiral. The only exceptions are the $N = 2$ hypermultiplets - for these, the previous argument doesn't work because they do not include $\lambda = \pm 1$ states, but since $\lambda = \frac{1}{2}$- and $\lambda = -\frac{1}{2}$ states are in the same multiplet, there can't be chirality either in this multiplet. Therefore only $N = 1, 0$ can be chiral, for instance $N = 1$ with $\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$ predicting at least one extra particle for each Standard Model particle. These particles have not been observed, however. Therefore the only hope for a realistic supersymmetric theory is: broken $N = 1$ supersymmetry at low energies $E \approx 10^2$ GeV.

2.5.3 Massive representations of $\mathcal{N} > 1$ supersymmetry and BPS states

Now consider $p_\mu = (m,\ 0,\ 0,\ 0)$, so

$$\left\{ Q^A_\alpha \ , \ \bar{Q}_{\dot{\beta} B} \right\} \ = \ 2\, m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta^A{}_B \ .$$

Contrary to the massless case, here the central charges can be non-vanishing. Therefore we have to distinguish two cases:

- $Z^{AB} = 0$

  There are $2\mathcal{N}$ creation- and annihilation operators

  $$a^A_\alpha \ := \ \frac{Q^A_\alpha}{\sqrt{2m}} \ , \qquad a^{A\dagger}_{\dot{\alpha}} \ := \ \frac{\bar{Q}^A_{\dot{\alpha}}}{\sqrt{2m}}$$

  leading to 2^{2N} states, each of them with dimension $(2y + 1)$. In the $\mathcal{N} = 2$ case, we find:

  $$\begin{array}{rc} |\Omega\rangle & 1 \times \text{spin } 0 \\ a^{A\dagger}_{\dot{\alpha}}\, |\Omega\rangle & 4 \times \text{spin } \frac{1}{2} \\ a^{A\dagger}_{\dot{\alpha}}\, a^{B\dagger}_{\dot{\beta}}\, |\Omega\rangle & 3 \times \text{spin } 0\ ,\ 3 \times \text{spin } 1 \\ a^{A\dagger}_{\dot{\alpha}}\, a^{B\dagger}_{\dot{\beta}}\, a^{C\dagger}_{\dot{\gamma}}\, |\Omega\rangle & 4 \times \text{spin } \frac{1}{2} \\ a^{A\dagger}_{\dot{\alpha}}\, a^{B\dagger}_{\dot{\beta}}\, a^{C\dagger}_{\dot{\gamma}}\, a^{D\dagger}_{\dot{\delta}}\, |\Omega\rangle & 1 \times \text{spin } 0 \end{array} \ ,$$

  i.e. as predicted $16 = 2^4$ states in total. Notice that these multiplets are much larger than the massless ones with only $2^{\mathcal{N}}$ states, due to the fact that in that case, half of the supersymmetry generators vanish ($Q^A_2 = 0$).

- $Z^{AB} \neq 0$

  Define the scalar quantity $\mathcal{H}$ to be (again, implicitly sandwiching in a bra/ket)

$$\mathcal{H} \ := \ (\bar{\sigma}^0)^{\dot{\beta}\alpha} \Big\{ Q^A_\alpha \ - \ \Gamma^A_\alpha \ , \ \bar{Q}_{\dot{\beta}A} \ - \ \bar{\Gamma}_{\dot{\beta}A} \Big\} \ \geq \ 0 \ .$$

  As a sum of products $AA^\dagger$, $\mathcal{H}$ is positive semi-definite, and the Γ^A_α are defined as

$$\Gamma^A_\alpha \ := \ \epsilon_{\alpha\beta} \, U^{AB} \, \bar{Q}_{\dot{\gamma}B} \, (\bar{\sigma}^0)^{\dot{\gamma}\beta}$$

  for some unitary matrix U (satisfying $UU^\dagger = \mathbb{1}$). We derive

$$\mathcal{H} \ = \ 8\, m \mathcal{N} \ - \ 2 \, \mathrm{Tr}\Big\{ Z\, U^\dagger \ + \ U\, Z^\dagger \Big\} \ \geq \ 0 \ .$$

  Due to the polar decomposition theorem, each matrix Z can be written as a product $Z = HV$ of a positive semi-definite hermitian matrix $H = H^\dagger$ and a unitary phase matrix $V = (V^\dagger)^{-1}$. Choosing $U = V$,

$$\mathcal{H} \ = \ 8\, m \mathcal{N} \ - \ 4 \, \mathrm{Tr}\Big\{ H \Big\} \ = \ 8\, m \mathcal{N} \ - \ 4 \, \mathrm{Tr}\Big\{ \sqrt{Z^\dagger Z} \Big\} \ \geq \ 0 \ .$$

  This is the BPS - bound for the mass m:

$$\boxed{ m \ \geq \ \frac{1}{2\mathcal{N}} \ \mathrm{Tr}\Big\{ \sqrt{Z^\dagger Z} \Big\} }$$

  States of minimal $m = \frac{1}{2\mathcal{N}} \mathrm{Tr}\Big\{ \sqrt{Z^\dagger Z} \Big\}$ are called *BPS states* (due to Bogomolnyi, Prasad and Sommerfeld). They are characterised by a vanishing combination $\bar{Q}^A_{\dot{\alpha}} - \bar{\Gamma}^A_{\dot{\alpha}}$, so the multiplet is shorter (similar to the massless case in which $Q^a_2 = 0$) having only $2^{\mathcal{N}}$ instead of $2^{2\mathcal{N}}$ states.

  For $\mathcal{N} = 2$, we define the components of the antisymmetric Z^{AB} to be

$$Z^{AB} \ = \ \begin{pmatrix} 0 & q_1 \\ -q_1 & 0 \end{pmatrix} \ \Longrightarrow \ m \ \geq \ \frac{q_1}{2} \ .$$

  More generally, if $\mathcal{N} > 2$ (but $\mathcal{N}$ even) we may perform a similarity transform[7] such that

$$Z^{AB} \ = \ \begin{pmatrix} 0 & q_1 & 0 & 0 & 0 & \cdots & & \\ -q_1 & 0 & 0 & 0 & 0 & \cdots & & \\ 0 & 0 & 0 & q_2 & 0 & \cdots & & \\ 0 & 0 & -q_2 & 0 & 0 & \cdots & & \\ 0 & 0 & 0 & 0 & \ddots & & & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & & \\ & & & & & & 0 & q_{\frac{\mathcal{N}}{2}} \\ & & & & & & -q_{\frac{\mathcal{N}}{2}} & 0 \end{pmatrix} , \tag{2.10}$$

[7] If $\mathcal{N} > 2$ but $\mathcal{N}$ is odd, we obtain Eq. 2.10 with the block matrices extending to $q_{(\mathcal{N}-1)/2}$ and an extra column and row of zeroes.

the BPS conditions hold block by block: $m \geq \frac{1}{2}\max_i(q_i)$, since we could define one $\mathcal{H}$ for each block. If k of the q_i are equal to $2m$, there are $2\mathcal{N} - 2k$ creation operators and $2^{2(\mathcal{N}-k)}$ states.

$$\begin{array}{rcl} k & = & 0 \quad \Longrightarrow \quad 2^{2\mathcal{N}} \text{ states, long multiplet} \\ 0 \; < \; k & < & \dfrac{\mathcal{N}}{2} \quad \Longrightarrow \quad 2^{2(\mathcal{N}-k)} \text{ states, short multiplets} \\ k & = & \dfrac{\mathcal{N}}{2} \quad \Longrightarrow \quad 2^{\mathcal{N}} \text{ states, ultra - short multiplet} \end{array}$$

Let us conclude this section about non-vanishing central charges with some remarks:

(i) BPS states and bounds came from *soliton* (monopole-) solutions of Yang Mills systems, which are localised finite energy solutions of the classical equations of motion. The bound refers to an energy bound.

(ii) The BPS states are stable since they are the lightest centrally charged particles.

(iii) Extremal black holes (which are the end points of the Hawking evaporation and therefore stable) happen to be BPS states for extended supergravity theories. Indeed, the equivalence of mass and charge reminds us of charged black holes.

(iv) BPS states are important in understanding strong-weak coupling dualities in field- and string theory.

(v) In string theory extended objects known as *D branes* are BPS.

3 Superspace and Superfields

So far, we have just considered 1 particle states in supermultiplets. Our goal is to arrive at a supersymmetric field theory describing interactions. Recall that particles are described by fields $\varphi(x^\mu)$ with the properties:

- they are functions of the coordinates x^μ in Minkowski space-time
- φ transforms under the Poincaré group

In the supersymmetric case, we want to deal with objects $\Phi(X)$ which

- are function of coordinates X of superspace
- transform under the super Poincaré group.

But what is that superspace?

3.1 Basics about superspace

3.1.1 Groups and cosets

We know that every continuous group G defines a manifold $\mathcal{M}_G$ via its parameters $\{\alpha_a\}$

$$\Lambda : \; G \;\longrightarrow\; \mathcal{M}_G \;, \qquad \Big\{ g = \exp(i\alpha_a T^a) \Big\} \;\longrightarrow\; \Big\{ \alpha_a \Big\} \;,$$

where $\dim G = \dim \mathcal{M}_G$. Consider for example:

- $G = U(1)$ with elements $g = \exp(i\alpha Q)$, then $\alpha \in [0, 2\pi]$, so the corresponding manifold is the 1 - sphere (a circle) $\mathcal{M}_{U(1)} = S^1$.

- $G = SU(2)$ with elements $g = \left(\begin{smallmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{smallmatrix}\right)$, where complex parameters α and β satisfy $|\alpha|^2 + |\beta|^2 = 1$. Write $\alpha = x_1 + ix_2$ and $\beta = x_3 + ix_4$ for $x_k \in \mathbb{R}$, then the constraint for p, q implies $\sum_{k=1}^4 x_k^2 = 1$, so $\mathcal{M}_{SU(2)} = S^3$

- $G = SL(2, \mathbb{C})$ with elements $g = e^a \cdot V$, $V \in SU(2)$ and a is traceless and hermitian, i.e.
$$a = \begin{pmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & -x_1 \end{pmatrix}$$
for $x_i \in \mathbb{R}$, so $\mathcal{M}_{SL(2,\mathbb{C})} = \mathbb{R}^3 \times S^3$.

To be more general, let's define a coset G/H where $g \in G$ is identified with $g{\cdot}h \,\forall\, h \in H \subset G$, e.g.

- $G = U_1(1) \times U_2(1) \ni g = \exp\big(i(\alpha_1 Q_1 + \alpha_2 Q_2)\big)$, $H = U_1(1) \ni h = \exp(i\beta Q_1)$. In $G/H = \big(U_1(1) \times U_2(1)\big)/U_1(1)$, the identification is
$$g\, h \ = \ \exp\Big\{ i \left((\alpha_1 \ + \ \beta)\, Q_1 \ + \ \alpha_2\, Q_2 \right) \Big\} \ = \ \exp\big(i\, (\alpha_1\, Q_1 \ + \ \alpha_2\, Q_2) \big) \ = \ g \ ,$$
so only α_2 contains effective information, $G/H = U_2(1)$.

- $G/H = SU(2)/U(1) \cong SO(3)/SO(2)$: Each $g \in SU(2)$ can be written as $g = \left(\begin{smallmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{smallmatrix}\right)$, identifying this by a $U(1)$ element $\mathrm{diag}(e^{i\gamma}, e^{-i\gamma})$ makes α effectively real. Hence, the parameter space is the 2 sphere ($\beta_1^2 + \beta_2^2 + \alpha^2 = 1$), i.e. $\mathcal{M}_{SU(2)/U(1)} = S^2$.

- More generally, $\mathcal{M}_{SO(n+1)/SO(n)} = S^n$.

- Minkowski = Poincaré / Lorentz = $\{\omega^{\mu\nu}, a^\mu\}/\{\omega^{\mu\nu}\}$ simplifies to the translations $\{a^\mu = x^\mu\}$ which can be identified with Minkowski space.

We define $\mathcal{N} = 1$ *superspace* to be the coset
$$\text{Super Poincaré / Lorentz} \ = \ \Big\{ \omega^{\mu\nu}, a^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}} \Big\} \ / \ \Big\{ \omega^{\mu\nu} \Big\} \ .$$

Recall that the general element g of super Poincaré group is given by
$$g \ = \ \exp\big(i\, (\omega^{\mu\nu}\, M_{\mu\nu} \ + \ a^\mu\, P_\mu \ + \ \theta^\alpha\, Q_\alpha \ + \ \bar{\theta}_{\dot{\alpha}}\, \bar{Q}^{\dot{\alpha}}) \big) \ ,$$
where Grassmann parameters θ^α, $\bar{\theta}_{\dot{\beta}}$ reduce anticommutation relations for Q_α, $\bar{Q}^{\dot{\beta}}$ to commutators because $\{Q_\alpha, \bar{\theta}_{\dot{\beta}}\} = \{\bar{Q}_{\dot{\alpha}}, \theta_\beta\} = 0$:
$$\Big\{ Q_\alpha \ , \ \bar{Q}_{\dot{\alpha}} \Big\} \ = \ 2\, (\sigma^\mu)_{\alpha\dot{\alpha}}\, P_\mu \quad \Longrightarrow \quad \Big[\theta Q \ , \ \bar{\theta}\, \bar{Q} \Big] \ = \ 2\, \theta^\alpha\, (\sigma^\mu)_{\alpha\dot{\beta}}\, \bar{\theta}^{\dot{\beta}}\, P_\mu .$$

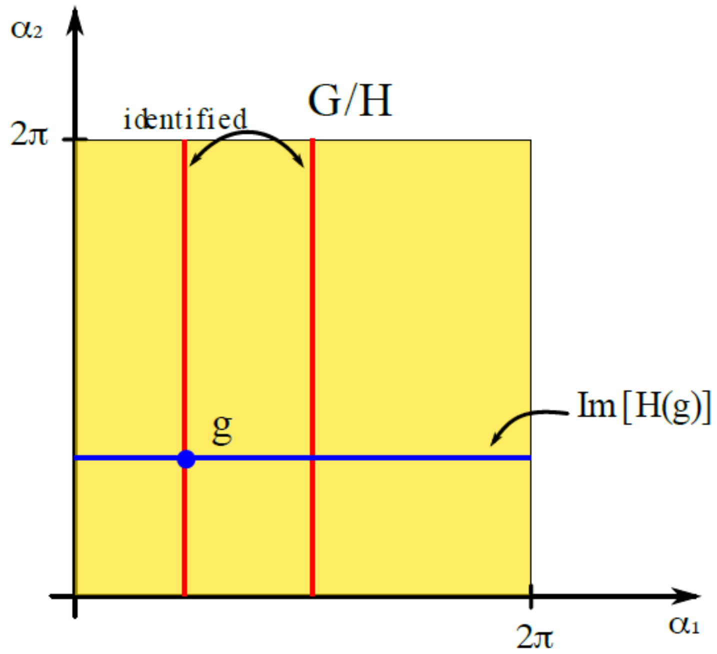


Figure 3. Illustration of the coset identity $G/H = \big(U_1(1) \times U_2(1)\big)/U_1(1) = U_2(1)$: The blue horizontal line shows the orbit of some $G = U_1(1) \times U_2(1)$ element g under the $H = U_1(1)$ group which is divided out. All its points are identified in the coset. Any red (dark) vertical line contains all the distinct coset elements and is identified with its neighbours in α_1 direction.

3.1.2 Properties of Grassmann variables

Superspace was first introduced in 1974 by Salam and Strathdee [6, 7]. Recommendable books about this subject are [8] and [9].

Let us first consider one single variable η. When trying to expand a generic (analytic) function in η as a power series, the fact that η squares to zero, $\eta^2 = 0$, cancels all the terms except for two,

$$f(\eta) \;=\; \sum_{k=0}^{\infty} f_k\,\eta^k \;=\; f_0 \;+\; f_1\,\eta \;+\; f_2\,\underbrace{\eta^2}_{0} + \underbrace{\ldots}_{0} \;=\; f_0 \;+\; f_1\,\eta \;.$$

So the most general function $f(\eta)$ is linear. Of course, its derivative is given by $\frac{\mathrm{d}f}{\mathrm{d}\eta} = f_1$. For integrals, define

$$\int \mathrm{d}\eta\, \frac{\mathrm{d}f}{\mathrm{d}\eta} \;:=\; 0 \quad\Longrightarrow\quad \int \mathrm{d}\eta \;=\; 0 \;,$$

as if there were no boundary terms. For integrals over η, we define

$$\int \mathrm{d}\eta\; \eta \;:=\; 1 \quad\Longrightarrow\quad \delta(\eta) \;=\; \eta \;.$$

The integral over a function $f(\eta)$ is then equal to its derivative,

$$\int \mathrm{d}\eta \; f(\eta) \;\; = \;\; \int \mathrm{d}\eta \; (f_0 \; + \; f_1 \, \eta) \;\; = \;\; f_1 \;\; = \;\; \frac{\mathrm{d}f}{\mathrm{d}\eta} \; .$$

Next, let θ^α, $\bar{\theta}_{\dot{\alpha}}$ be spinors of Grassmann numbers. Their squares are defined by

$$\theta\theta \;\; := \;\; \theta^\alpha \, \theta_\alpha \; , \qquad \bar{\theta}\bar{\theta} \;\; := \;\; \bar{\theta}_{\dot{\alpha}} \, \bar{\theta}^{\dot{\alpha}}$$

$$\Longrightarrow \quad \theta^\alpha \, \theta^\beta \;\; = \;\; -\frac{1}{2} \, \epsilon^{\alpha\beta} \, \theta\theta \; , \qquad \bar{\theta}^{\dot{\alpha}} \, \bar{\theta}^{\dot{\beta}} \;\; = \;\; \frac{1}{2} \, \epsilon^{\dot{\alpha}\dot{\beta}} \, \bar{\theta}\bar{\theta} \; .$$

Derivatives work in analogy to Minkowski coordinates:

$$\partial_\alpha \theta^\beta := \frac{\partial \theta^\beta}{\partial \theta^\alpha} \;\; = \;\; \delta_\alpha{}^\beta \quad \Longrightarrow \quad \bar{\partial}_{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} := \frac{\partial \bar{\theta}^{\dot{\beta}}}{\partial \bar{\theta}^{\dot{\alpha}}} \;\; = \;\; \delta_{\dot{\alpha}}{}^{\dot{\beta}}$$

where $\{\partial_\alpha, \; \partial_\beta\} = \{\bar{\partial}_{\dot{\alpha}}, \; \bar{\partial}_{\dot{\beta}}\} = 0$. As for multi-dimensional integrals,

$$\int \mathrm{d}\theta^1 \int \mathrm{d}\theta^2 \; \theta^2 \, \theta^1 \;\; = \;\; \frac{1}{2} \int \mathrm{d}\theta^1 \int \mathrm{d}\theta^2 \; \theta\theta \;\; = \;\; 1 \; ,$$

which justifies the definition

$$\int \mathrm{d}^2\theta := \frac{1}{2} \int \mathrm{d}\theta^1 \int \mathrm{d}\theta^2 \;\; \Rightarrow \int \mathrm{d}^2\theta \; \theta\theta \;\; = \;\; 1 \text{ and} \int \mathrm{d}^2\theta \int \mathrm{d}^2\bar{\theta} \; (\theta\theta) \, (\bar{\theta}\bar{\theta}) \;\; = \;\; 1 \; .$$

Note that $\int 1 \; d\theta^\alpha = \int 1 \; d\bar{\theta}^{\dot{\alpha}} = 0$. Also, written in terms of ϵ:

$$\mathrm{d}^2\theta \;\; = \;\; -\frac{1}{4} \, \mathrm{d}\theta^\alpha \, \mathrm{d}\theta^\beta \, \epsilon_{\alpha\beta} \; , \qquad \mathrm{d}^2\bar{\theta} \;\; = \;\; \frac{1}{4} \, \mathrm{d}\bar{\theta}^{\dot{\alpha}} \, \mathrm{d}\bar{\theta}^{\dot{\beta}} \, \epsilon_{\dot{\alpha}\dot{\beta}} \; .$$

or

$$d^2\theta = \frac{1}{4} \epsilon_{\beta\alpha} d\theta^\alpha d\theta^\beta, \qquad d^2\bar{\theta} = -\frac{1}{4} \epsilon_{\dot{\alpha}\dot{\beta}} d\bar{\theta}^{\dot{\beta}} d\bar{\theta}^{\dot{\alpha}}.$$

3.1.3 Definition and transformation of the general scalar superfield

To define a *superfield*, recall properties of scalar fields $\varphi(x^\mu)$:

- function of space-time coordinates x^μ
- transformation under Poincaré

Treating φ as an operator, a translation with parameter a_μ will change it to

$$\varphi \;\; \mapsto \;\; \exp(-ia_\mu \, P^\mu) \, \varphi \, \exp(ia_\mu \, P^\mu) \; . \tag{3.1}$$

But $\varphi(x^\mu)$ is also a Hilbert vector in some function space $\mathcal{F}$, so

$$\varphi(x^\mu) \;\; \mapsto \;\; \exp(-ia_\mu \, \mathcal{P}^\mu) \, \varphi(x^\mu) \;\; =: \;\; \varphi(x^\mu \, - \, a^\mu) \quad \Longrightarrow \quad \mathcal{P}_\mu \;\; = \;\; -i\partial_\mu \; . \tag{3.2}$$

$\mathcal{P}^\mu$ is a representation of the abstract operator P^μ acting on $\mathcal{F}$. Comparing the two transformation rules Eqs. 3.1,3.2 to first order in a_μ, we get the following relationship:

$$\left(1 - ia_\mu \, P^\mu\right) \varphi \left(1 + ia_\mu \, P^\mu\right) \;\; = \;\; \left(1 - ia_\mu \, \mathcal{P}^\mu\right) \varphi \Leftrightarrow i \left[\varphi \, , \, a_\mu \, P^\mu\right] \;\; = \;\; -ia^\mu \, \mathcal{P}_\mu \, \varphi \;\; = \;\; -a^\mu \, \partial_\mu \, \varphi.$$

We shall perform a similar (but super-) transformation on a superfield.

For a general scalar superfield $S(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$, one can perform an expansion in powers of θ_α, $\bar{\theta}_{\dot{\alpha}}$ with a finite number of nonzero terms:

$$\begin{aligned} S(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) \;&=\; \varphi(x) \;+\; \theta\psi(x) \;+\; \bar{\theta}\bar{\chi}(x) \;+\; \theta\theta\, M(x) \;+\; \bar{\theta}\bar{\theta}\, N(x) \;+\; (\theta\,\sigma^\mu\,\bar{\theta})\, V_\mu(x) \\ &\quad+\; (\theta\theta)\,\bar{\theta}\bar{\lambda}(x) \;+\; (\bar{\theta}\bar{\theta})\,\theta\rho(x) \;+\; (\theta\theta)\,(\bar{\theta}\bar{\theta})\, D(x) \end{aligned} \tag{3.3}$$

Question: Why is there no term $(\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta})F_{\mu\nu}$?

We have the transformation of $S(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$ under the super Poincaré group, firstly as a field operator

$$S(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) \;\mapsto\; \exp\bigl(-i\,(\epsilon Q \,+\, \bar{\epsilon}\bar{Q})\bigr)\, S\, \exp\bigl(i\,(\epsilon Q \,+\, \bar{\epsilon}\bar{Q})\bigr)\ , \tag{3.4}$$

secondly as a Hilbert vector

$$S(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) \;\mapsto\; \exp\bigl(i\,(\epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}})\bigr)\, S(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) \;=\; S\bigl(x^\mu + \delta x^\mu,\, \theta_\alpha + \epsilon_\alpha,\, \bar{\theta}_{\dot{\alpha}} + \bar{\epsilon}_{\dot{\alpha}}\bigr)\ . \tag{3.5}$$

Here, ϵ denotes a parameter, $\mathcal{Q}$ a representation of the spinorial generators Q_α acting on functions of θ, $\bar{\theta}$, and c is a constant to be fixed later, which is involved in the translation

$$\delta x^\mu = -\, ic\,(\epsilon\,\sigma^\mu\,\bar{\theta}) \;+\; ic^*\,(\theta\,\sigma^\mu\,\bar{\epsilon})\ .$$

The translation of arguments x^μ, θ_α, $\bar{\theta}_{\dot{\alpha}}$ implies,

$$\boxed{\begin{aligned} \mathcal{Q}_\alpha \;&=\; -i\,\frac{\partial}{\partial\theta^\alpha} \;-\; c\,(\sigma^\mu)_{\alpha\dot{\beta}}\,\bar{\theta}^{\dot{\beta}}\,\frac{\partial}{\partial x^\mu} \\ \bar{\mathcal{Q}}_{\dot{\alpha}} \;&=\; +i\,\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \;+\; c^*\,\theta^\beta\,(\sigma^\mu)_{\beta\dot{\alpha}}\,\frac{\partial}{\partial x^\mu} \\ \mathcal{P}_\mu \;&=\; -i\partial_\mu\ , \end{aligned}}$$

where c can be determined from the commutation relation which, of course, holds in any representation:

$$\Bigl\{\mathcal{Q}_\alpha\ ,\ \bar{\mathcal{Q}}_{\dot{\alpha}}\Bigr\} \;=\; 2\,(\sigma^\mu)_{\alpha\dot{\alpha}}\,\mathcal{P}_\mu \quad\Longrightarrow\quad \mathrm{Re}\{c\} \;=\; 1$$

It is convenient to set $c = 1$. Again, a comparison of the two expressions (to first order in ϵ) for the transformed superfield S is the key to get its commutation relations with Q_α:

$$\boxed{i\,\Bigl[S\ ,\ \epsilon Q \,+\, \bar{\epsilon}\bar{Q}\Bigr] \;=\; i\,\bigl(\epsilon\mathcal{Q} \,+\, \bar{\epsilon}\bar{\mathcal{Q}}\bigr)\,S \;=\; \delta S}$$

Considering an infinitesimal transformation $S \to S + \delta S = (1 + i\epsilon\mathcal{Q} + i\bar{\epsilon}\bar{\mathcal{Q}})S$, where

$$\begin{aligned} \delta S \;:=\; &\delta\varphi(x) \;+\; \theta\delta\psi(x) \;+\; \bar{\theta}\delta\bar{\chi}(x) \;+\; \theta\theta\,\delta M(x) \;+\; \bar{\theta}\bar{\theta}\,\delta N(x) \;+\; (\theta\,\sigma^\mu\,\bar{\theta})\,\delta V_\mu(x) \\ &+\; (\theta\theta)\,\bar{\theta}\delta\bar{\lambda}(x) \;+\; (\bar{\theta}\bar{\theta})\,\theta\delta\rho(x) \;+\; (\theta\theta)\,(\bar{\theta}\bar{\theta})\,\delta D(x). \end{aligned} \tag{3.6}$$

Substituting for $\mathcal{Q}_\alpha$, $\bar{\mathcal{Q}}_{\dot\alpha}$ and S, we get explicit terms for the changes in the different parts of S:

$$\begin{aligned}
\delta\varphi &= \epsilon\psi + \bar\epsilon\bar\chi, & \delta\psi &= 2\epsilon M + (\sigma^\mu\bar\epsilon)(i\partial_\mu\varphi + V_\mu)\\
\delta\bar\chi &= 2\bar\epsilon N - (\epsilon\sigma^\mu)(i\partial_\mu\varphi - V_\mu) & \delta M &= \bar\epsilon\bar\lambda - \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar\epsilon\\
\delta V_\mu &= \epsilon\sigma_\mu\bar\lambda + \rho\sigma_\mu\bar\epsilon + \frac{i}{2}\left(\partial^\nu\psi\sigma_\mu\bar\sigma_\nu\epsilon - \bar\epsilon\bar\sigma_\nu\sigma_\mu\partial^\nu\bar\chi\right) & \delta N &= \epsilon\rho + \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\bar\chi\\
\delta\bar\lambda &= 2\bar\epsilon D + \frac{i}{2}\left(\bar\sigma^\nu\sigma^\mu\bar\epsilon\right)\partial_\mu V_\nu + i(\bar\sigma^\mu\epsilon)\partial_\mu M & \delta D &= \frac{i}{2}\partial_\mu(\epsilon\sigma^\mu\bar\lambda - \rho\sigma^\mu\bar\epsilon)\\
\delta\rho &= 2\epsilon D - \frac{i}{2}\left(\sigma^\nu\bar\sigma^\mu\epsilon\right)\partial_\mu V_\nu + i(\sigma^\mu\bar\epsilon)\partial_\mu N
\end{aligned}$$

as on the second examples sheet. Note that δD is a total derivative. Also, we have bosons and fermions transforming into each other).

3.1.4 Remarks on superfields

S is a superfield $\Leftrightarrow$ it satisfies $\delta S = i(\epsilon\mathcal{Q} + \bar\epsilon\bar{\mathcal{Q}})S$. Thus:

- If S_1 and S_2 are superfields then so is the product S_1S_2:

$$\begin{aligned}
\delta(S_1\,S_2) &= S_1\delta S_2 + (\delta S_1)S_2\\
&= S_1\left(i\,(\epsilon\mathcal{Q} + \bar\epsilon\bar{\mathcal{Q}})\,S_2\right) + \left(i\,(\epsilon\mathcal{Q} + \bar\epsilon\bar{\mathcal{Q}})\,S_1\right)S_2\\
&= i\,(\epsilon\mathcal{Q} + \bar\epsilon\bar{\mathcal{Q}})\,(S_1\,S_2)
\end{aligned} \tag{3.7}$$

  In the last step, we used the Leibnitz property of the $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ as differential operators.

- Linear combinations of superfields are superfields again (straightforward proof).

- $\partial_\mu S$ is a superfield but $\partial_\alpha S$ is not:

$$\delta(\partial_\alpha S) = \partial_\alpha(\delta S) = i\partial_\alpha[(\epsilon\mathcal{Q} + \bar\epsilon\bar{\mathcal{Q}})S] \neq i(\epsilon\mathcal{Q} + \bar\epsilon\bar{\mathcal{Q}})\,(\partial_\alpha S)$$

  since $[\partial_\alpha, \epsilon\mathcal{Q} + \bar\epsilon\bar{\mathcal{Q}}] \neq 0$. We need to define a covariant derivative,

$$\mathcal{D}_\alpha := \partial_\alpha + i(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\,\partial_\mu\,, \qquad \bar{\mathcal{D}}_{\dot\alpha} := -\bar\partial_{\dot\alpha} - i\theta^\beta\,(\sigma^\mu)_{\beta\dot\alpha}\,\partial_\mu$$

  which satisfies

$$\left\{\mathcal{D}_\alpha\,,\,\mathcal{Q}_\beta\right\} = \left\{\mathcal{D}_\alpha\,,\,\bar{\mathcal{Q}}_{\dot\beta}\right\} = \left\{\bar{\mathcal{D}}_{\dot\alpha}\,,\,\mathcal{Q}_\beta\right\} = \left\{\bar{\mathcal{D}}_{\dot\alpha}\,,\,\bar{\mathcal{Q}}_{\dot\beta}\right\} = 0$$

  and therefore

$$\left[\mathcal{D}_\alpha\,,\,\epsilon\mathcal{Q} + \bar\epsilon\bar{\mathcal{Q}}\right] = 0 \implies \mathcal{D}_\alpha S \quad \text{is superfield.}$$

  Also note that super-covariant derivatives satisfy anticommutation relations

$$\left\{\mathcal{D}_\alpha\,,\,\bar{\mathcal{D}}_{\dot\beta}\right\} = -2i\,(\sigma^\mu)_{\alpha\dot\beta}\,\partial_\mu\,, \qquad \left\{\mathcal{D}_\alpha\,,\,\mathcal{D}_\beta\right\} = \left\{\bar{\mathcal{D}}_{\dot\alpha}\,,\,\bar{\mathcal{D}}_{\dot\beta}\right\} = 0\,.$$

- $S = f(x)$ is a superfield only if $f = const$, otherwise, there would be some $\delta\psi \propto \epsilon\partial^\mu f$. For constant spinor c, $S = c\theta$ is not a superfield due to $\delta\phi = \epsilon c$.

S is **not** an irreducible representation of supersymmetry, so we can eliminate some of its components keeping it still as a superfield. In general we can impose consistent constraints on S, leading to smaller superfields that are irreducible representations of the supersymmetry algebra. There are different types depending upon the constraint:

- chiral superfield Φ such that $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0$
- anti-chiral superfield $\bar{\Phi}$ such that $\mathcal{D}_\alpha\bar{\Phi} = 0$
- vector (or real) superfield $V = V^\dagger$
- linear superfield L such that $\mathcal{D}\mathcal{D}L = 0$ and $L = L^\dagger$.

3.2 Chiral superfields

We want to find the components of a superfields Φ satisfying $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0$. We define for convenience

$$y^\mu \ := \ x^\mu \ + \ i\theta\,\sigma^\mu\,\bar{\theta}\ .$$

If $\Phi = \Phi(y, \theta, \bar{\theta})$, then, since $\bar{\mathcal{D}}_{\dot{\alpha}}$ is a differential operator,

$$\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = (\bar{\mathcal{D}}_{\dot{\alpha}}\theta^\alpha)\,\frac{\partial\Phi}{\partial\theta^\alpha}\bigg|_{y,\bar{\theta}} + (\bar{\mathcal{D}}_{\dot{\alpha}}y^\mu)\,\frac{\partial\Phi}{\partial y^\mu}\bigg|_{\theta,\bar{\theta}} + (\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}})\,\frac{\partial\Phi}{\partial\bar{\theta}^{\dot{\beta}}}\bigg|_{y,\theta}\ .$$

We have $(\bar{\mathcal{D}}_{\dot{\alpha}}\theta^\alpha) = 0$ and $(\bar{\mathcal{D}}_{\dot{\alpha}}y^\mu) = (-\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha\sigma^\rho_{\alpha\dot{\alpha}}\partial_\rho)(x^\mu + i\theta\sigma^\mu\bar{\theta}) = i(\theta\sigma^\mu)_{\dot{\alpha}} - i(\theta\sigma^\mu)_{\dot{\alpha}} = 0$, hence the chiral superfield condition becomes $\frac{\partial\Phi}{\partial\bar{\theta}^{\dot{\beta}}} = 0$. Thus there is no $\bar{\theta}^{\dot{\alpha}}$ - dependence and Φ depends only on y and θ. In components, one finds

$$\Phi(y^\mu, \theta^\alpha) \ = \ \varphi(y^\mu) \ + \ \sqrt{2}\,\theta\psi(y^\mu) \ + \ \theta\theta\,F(y^\mu)\ ,$$

where the left handed supercovariant derivative acts as $\mathcal{D}_\alpha = \partial_\alpha + 2i(\sigma^\mu\bar{\theta})_\alpha\frac{\partial}{\partial y^\mu}$ on $\Phi(y^\mu, \theta^\alpha)$. The physical components of a chiral superfield are as follows: φ represents a scalar part (squarks, sleptons, Higgs), ψ some $s = \frac{1}{2}$ particles (quarks, leptons, Higgsino) and F is an *auxiliary field* in a way to be defined later. Off shell, there are 4 bosonic (complex φ, F) and 4 fermionic (complex ψ_α) components. Performing a Taylor expansion of Φ around x^μ:

$$\boxed{\begin{aligned}\Phi(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) \ &= \ \varphi(x) \ + \ \sqrt{2}\,\theta\psi(x) \ + \ \theta\theta\,F(x) \ + \ i\theta\,\sigma^\mu\,\bar{\theta}\,\partial_\mu\varphi(x) \\ &\quad - \ \frac{i}{\sqrt{2}}\,(\theta\theta)\,\partial_\mu\psi(x)\,\sigma^\mu\,\bar{\theta} \ - \ \frac{1}{4}\,(\theta\theta)\,(\bar{\theta}\bar{\theta})\,\partial_\mu\partial^\mu\varphi(x)\end{aligned}}$$

Under a supersymmetry transformation

$$\delta\Phi \ = \ i\left(\epsilon\mathcal{Q} \ + \ \bar{\epsilon}\bar{\mathcal{Q}}\right)\Phi\ ,$$

we find for the change in components

$$\boxed{\begin{aligned} \delta\varphi &= \sqrt{2}\,\epsilon\psi \\ \delta\psi &= i\sqrt{2}\,\sigma^\mu\,\bar{\epsilon}\,\partial_\mu\varphi \;+\; \sqrt{2}\,\epsilon\,F \\ \delta F &= i\sqrt{2}\,\bar{\epsilon}\,\bar{\sigma}^\mu\,\partial_\mu\psi\ . \end{aligned}}$$

So δF is another total derivative term, just like δD in a general superfield. Note that:

- The product of chiral superfields is a chiral superfield, since $\bar{\mathcal{D}}_{\dot{\alpha}}(S_1S_2) = (\bar{\mathcal{D}}_{\dot{\alpha}}S_1)S_2 + S_1\bar{\mathcal{D}}_{\dot{\alpha}}S_2 = 0$ if $\bar{\mathcal{D}}_{\dot{\alpha}}S_i = 0$. In general, any *holomorphic* function $f(\Phi)$ of a chiral superfield Φ is a chiral superfield.
- If Φ is chiral, then $\bar{\Phi} = \Phi^\dagger$ is anti-chiral.
- $\Phi^\dagger\Phi$ and $\Phi^\dagger + \Phi$ are real superfields but neither chiral nor anti-chiral.

4 Four dimensional supersymmetric Lagrangians

4.1 $\mathcal{N} = 1$ global supersymmetry

We want to determine couplings among superfields which include the particles of the Standard Model. For this we need a prescription to build Lagrangians which are invariant (up to a total derivative) under a supersymmetry transformation. We will start with the simplest case of only chiral superfields.

4.1.1 Chiral superfield Lagrangian

In order to find an object $\mathcal{L}(\Phi)$ such that $\delta\mathcal{L}$ is a total derivative under a supersymmetry transformation, we exploit that:

- For a general scalar superfield $S = ... + (\theta\theta)(\bar{\theta}\bar{\theta})D(x)$, the D term transforms as:

$$\delta D \;=\; \frac{i}{2}\,\partial_\mu\left(\epsilon\,\sigma^\mu\,\bar{\lambda} \;-\; \rho\,\sigma^\mu\,\bar{\epsilon}\right).$$

- For a chiral superfield $\Phi = ... + (\theta\theta)F(x)$, the F term transforms as:

$$\delta F \;=\; i\sqrt{2}\,\bar{\epsilon}\,\bar{\sigma}^\mu\,\partial_\mu\psi.$$

Since δF and δD are total derivatives, they have no effect on local physics in the action, and integrate to zero. For a chiral superfield $\Phi = \ldots + (\theta\theta)F$, thus the '$F-$term' $\Phi|_F$ is defined to be whatever multiplies $(\theta\theta)$. Thus, for example, under a SUSY transformation, $\int d^4x\Phi|_F = \int d^4xF \to \int d^4x(F + \delta F) = \int d^4xF$ is invariant. Therefore, the most general Lagrangian for a chiral superfield Φ's can be written as:

$$\mathcal{L} \;=\; \underbrace{K(\Phi,\Phi^\dagger)}_{\text{Kähler - potential}}\Big|_D \;+\; \left(\underbrace{W(\Phi)}_{\text{super - potential}}\Big|_F + h.c.\right).$$

Where $|_D$ refers to the D term of the corresponding superfield (whatever multiplies $(\bar\theta\bar\theta)(\theta\theta)$. The function K is known as the *Kähler potential*, a real function of Φ and $\Phi^\dagger$. $W(\Phi)$ is known as the *superpotential*, a holomorphic function of the chiral superfield Φ (and therefore is a chiral superfield itself).
In order to obtain a renormalisable theory, we need to construct a Lagrangian in terms of operators of dimensionality such that the Lagrangian has dimensionality 4. We know $[\varphi] = 1$ (where the square brackets stand for dimensionality of the field) and we want $[\mathcal{L}] = 4$. Terms of dimension 4, such as $\partial^\mu\varphi\partial_\mu\varphi^*$, $m^2\varphi\varphi^*$ and $g|\varphi|^4$, are renormalisable, but couplings with negative mass dimensions are not. The mass dimension of the superfield Φ is the same as that of its scalar component and the dimension of ψ is the same as any standard fermion, that is

$$[\Phi] \;=\; [\varphi] \;=\; 1\,, \qquad [\psi] \;=\; \frac{3}{2}$$

From the expansion $\Phi = \varphi + \sqrt{2}\theta\psi + \theta\theta F + ...$ it follows that

$$[\theta] \;=\; -\frac{1}{2}\,, \qquad [F] \;=\; 2\,.$$

This already hints that F is not a standard scalar field. In order to have $[\mathcal{L}] = 4$ we need:

$$\begin{aligned} [K_D] \;&\leq\; 4 \;\text{ in }\; K \;=\; ... + \,(\theta\theta)\,(\bar\theta\bar\theta)\,K_D \\ [W_F] \;&\leq\; 4 \;\text{ in }\; W \;=\; ... + \,(\theta\theta)\,W_F \\ \Longrightarrow \;\; [K] \;&\leq\; 2\,, \qquad [W] \;\leq\; 3\,. \end{aligned}$$

A possible renormalisable term for K is $\Phi^\dagger\Phi$, but not $\Phi + \Phi^\dagger$ or $\Phi\Phi + \Phi^\dagger\Phi^\dagger$ since these contain no $D-$terms.
Therefore we are lead to the following general expressions for K and W:

$$K \;=\; \Phi^\dagger\,\Phi\,, \qquad W \;=\; \alpha \,+\, \lambda\,\Phi \,+\, \frac{m}{2}\,\Phi^2 \,+\, \frac{g}{3}\,\Phi^3\,,$$

whose Lagrangian is known as *Wess Zumino model*:

$$\mathcal{L}_{WZ} \;=\; \Phi^\dagger\,\Phi\Big|_D \,+\, \left(W(\Phi)\Big|_F \,+\, h.c.\right). \tag{4.1}$$

We get the expression for $\Phi^\dagger\Phi\Big|_D$ by substituting

$$\Phi \;=\; \varphi \,+\, \sqrt{2}\,\theta\psi \,+\, \theta\theta\,F \,+\, i\theta\,\sigma^\mu\,\bar\theta\,\partial_\mu\varphi \,-\, \frac{i}{\sqrt{2}}\,(\theta\theta)\,\partial_\mu\psi\,\sigma^\mu\,\bar\theta \,-\, \frac{1}{4}\,(\theta\theta)\,(\bar\theta\bar\theta)\,\partial_\mu\partial^\mu\varphi. \tag{4.2}$$

We also perform a Taylor expansion around $\Phi = \varphi$ (where $\frac{\partial W}{\partial\varphi} = \frac{\partial W}{\partial\Phi}\Big|_{\Phi=\varphi}$):

$$W(\Phi) \;=\; W(\varphi) \,+\, \underbrace{(\Phi \,-\, \varphi)}_{...\,+\,\theta\theta F\,+\,...}\,\frac{\partial W}{\partial\varphi} \,+\, \underbrace{\frac{1}{2}\,(\Phi \,-\, \varphi)^2}_{...\,+\,(\theta\psi)\,(\theta\psi)\,+\,...}\,\frac{\partial^2 W}{\partial\varphi^2} \tag{4.3}$$

Substituting Eqs. 4.3,4.2 into Eq. 4.1, we obtain

$$\mathcal{L}_{WZ} = \partial^\mu \varphi^* \, \partial_\mu \varphi - i\bar{\psi}\, \bar{\sigma}^\mu \, \partial_\mu \psi + F\, F^* + \left(\frac{\partial W}{\partial \varphi}\; F + h.c. \right) - \frac{1}{2}\; \left(\frac{\partial^2 W}{\partial \varphi^2}\; \psi\psi + h.c. \right) .$$

The part of the Lagrangian depending on the 'auxiliary field' F takes the simple form:

$$\mathcal{L}_{(F)} \;\; = \;\; F\, F^* \; + \; \frac{\partial W}{\partial \varphi}\; F \; + \; \frac{\partial W^*}{\partial \varphi^*}\, F^*$$

Notice that this is quadratic and without any derivatives. This means that the field F does not propagate. Also, we can easily eliminate F using the field equations

$$\begin{aligned} \frac{\delta \mathcal{S}_{(F)}}{\delta F} \;&=\; 0 \quad \Longrightarrow \quad F^* \; + \; \frac{\partial W}{\partial \varphi} \;=\; 0 \\ \frac{\delta \mathcal{S}_{(F)}}{\delta F^*} \;&=\; 0 \quad \Longrightarrow \quad F \; + \; \frac{\partial W^*}{\partial \varphi^*} \;=\; 0 \end{aligned}$$

and substitute the result back into the Lagrangian,

$$\mathcal{L}_{(F)} \;\; \mapsto \;\; - \left| \frac{\partial W}{\partial \varphi} \right|^2 \;\; =: \;\; -V_{(F)}(\varphi) \; ,$$

This defines the scalar potential. From its expression we can easily see that it is a positive definite scalar potential $V_{(F)}(\varphi)$.
We finish the section about chiral superfield Lagrangian with two remarks,

- The $\mathcal{N} = 1$ Lagrangian is a particular case of standard $\mathcal{N} = 0$ Lagrangians: the scalar potential is positive semi-definite ($V \geq 0$). Also the mass for scalar field φ (as it can be read from the quadratic term in the scalar potential) equals the one for the spinor ψ (as can be read from the term $\frac{1}{2} \frac{\partial^2 W}{\partial \varphi^2} \psi\psi$). Moreover, the coefficient g of Yukawa coupling $g(\varphi\psi\psi)$ also determines the scalar self coupling, $g^2|\varphi|^4$. This is the source of some "miraculous" cancellations in SUSY perturbation theory: divergences are removed from some loop corrections, a la Fig. 4.

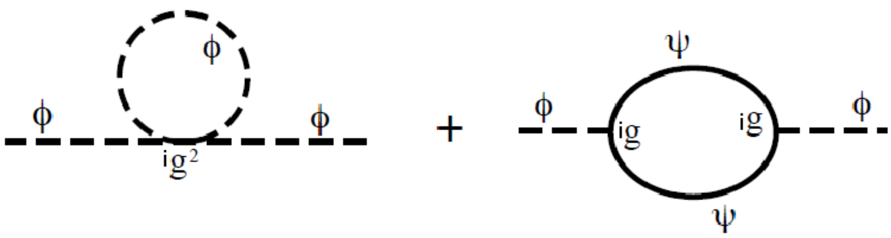


Figure 4. One loop diagrams which yield corrections to the scalar mass squared. SUSY relates the ϕ^4 coupling to the Yukawa couplings $\phi(\psi\bar{\psi})$ and therefore ensures cancellation of the leading divergence.

- In general, we may expand $K(\Phi_i, \Phi_j^\dagger)$ and $W(\Phi_i)$ around $\Phi_i = \varphi_i$ in components, from whence we get the kinetic terms, e.g.

$$K(\Phi_j^\dagger,\ \Phi_i)\big|_D = \ldots + \left(\frac{\partial^2 K}{\partial\varphi_i \partial\varphi_{\bar{j}}^*}\right) \partial_\mu\varphi_i\, \partial^\mu\varphi_{\bar{j}}^* \quad = \quad \ldots + K_{i\bar{j}}\, \partial_\mu\varphi_i\, \partial^\mu\varphi_{\bar{j}}^* \ .$$

  $K_{i\bar{j}}$ is a metric in a space which is a complex Kähler - manifold with coordinates φ_i.

4.2 Vector superfields

4.2.1 Definition and transformation of the vector superfield

The most general vector superfield $V(x,\theta,\bar{\theta}) = V^\dagger(x,\theta,\bar{\theta})$ has the form

$$\begin{aligned} V(x,\theta,\bar{\theta}) \ &= \ C(x) \ + \ i\theta\chi(x) \ - \ i\bar{\theta}\bar{\chi}(x) \ + \ \frac{i}{2}\,\theta\theta\,\big(M(x) \ + \ iN(x)\big) \ - \ \frac{i}{2}\,\bar{\theta}\bar{\theta}\,\big(M(x) \ - \ iN(x)\big) \\ &+ \ \theta\,\sigma^\mu\,\bar{\theta}\,V_\mu(x) \ + \ (\theta\theta)\,\bar{\theta}\left(\bar{\lambda}(x) \ - \ \frac{1}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right) \\ &+ \ (\bar{\theta}\bar{\theta})\,\theta\left(\lambda(x) \ + \ \frac{1}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right) \ + \ \frac{1}{2}\,(\theta\theta)\,(\bar{\theta}\bar{\theta})\,\left(D(x) \ - \ \frac{1}{2}\partial_\mu\partial^\mu C(x)\right) \ , \end{aligned}$$

where we have shifted some fields (notably D and λ) for convenience. There are 8 bosonic components C, M, N, D, V_μ and 4 + 4 fermionic ones ($\chi_\alpha,\ \lambda_\alpha$).
If Λ is a chiral superfield, then $i(\Lambda - \Lambda^\dagger)$ is a vector superfield. It has components:

$$\begin{aligned} C \ &= \ i\left(\varphi \ - \ \varphi^\dagger\right) \\ \chi \ &= \ \sqrt{2}\,\psi \\ \frac{1}{2}\,(M \ + \ iN) \ &= \ F \\ V_\mu \ &= \ -\partial_\mu\left(\varphi \ + \ \varphi^\dagger\right) \\ \lambda \ = \ D \ &= \ 0 \end{aligned}$$

Question: Can you derive these relations by substituting in for Λ?

We can define a generalised gauge transformations of vector fields via

$$V \ \mapsto \ V \ + i\left(\Lambda \ - \ \Lambda^\dagger\right) \ ,$$

which induces a standard gauge transformation for the vector component of V

$$V_\mu \ \mapsto \ V_\mu \ - \ \partial_\mu\left[\,2\mathrm{Re}(\varphi)\,\right] \ =: \ V_\mu \ + \ \partial_\mu\alpha \ .$$

Then we can choose φ, ψ, F within Λ to gauge away some of the components of V, as long as we have constructed a Lagrangian that is invariant under the generalised gauge transformation.

4.2.2 Wess Zumino gauge

We can choose the components of Λ above: φ, ψ, F in such a way to set $C = \chi = M = N = 0$. This defines the *Wess Zumino (WZ) gauge*, and we shall use this from now on. A vector superfield in Wess Zumino gauge reduces to the form

$$\boxed{V_{\rm WZ}(x,\theta,\bar\theta) \;=\; (\theta\,\sigma^\mu\,\bar\theta)\,V_\mu(x) \;+\; (\theta\theta)\,\big(\bar\theta\bar\lambda(x)\big) \;+\; (\bar\theta\bar\theta)\,\big(\theta\lambda(x)\big) \;+\; \frac{1}{2}\,(\theta\theta)\,(\bar\theta\bar\theta)\,D(x)\;.}$$

The physical components of a vector superfield are: V_μ corresponding to gauge particles (γ, $W^\pm$, Z, gluon), the λ and $\bar\lambda$ to gauginos and D is an auxiliary field, to be defined later. Powers of $V_{\rm WZ}$ are given by

$$V_{\rm WZ}^2 \;=\; \frac{1}{2}\,(\theta\theta)\,(\bar\theta\bar\theta)\,V^\mu\,V_\mu\;, \qquad V_{\rm WZ}^{2+n} \;=\; 0\;\forall\;n \in \mathbb{N}\;.$$

Note that the Wess Zumino gauge is not supersymmetric, since $V_{\rm WZ} \mapsto V'_{\cancel{\rm WZ}}$ under supersymmetry. However, under a combination of supersymmetry and generalised gauge transformation $V'_{\cancel{\rm WZ}} \mapsto V''_{\rm WZ}$ we can end up with a vector field in Wess Zumino gauge.

4.2.3 Abelian field strength superfield

Recall that a non-supersymmetric complex scalar field φ coupled to a gauge field V_μ via covariant derivative $D_\mu = \partial_\mu - iqV_\mu$ transforms like

$$\varphi(x) \;\mapsto\; \exp\big(iq\alpha(x)\big)\,\varphi(x)\;, \qquad V_\mu(x) \;\mapsto\; V_\mu(x) \;+\; \partial_\mu\alpha(x)$$

under local $U(1)$ with charge q and local parameter $\alpha(x)$.
Under supersymmetry, these concepts generalise to chiral superfields Φ and vector superfields V. To construct a gauge invariant quantity out of Φ and V, we impose the following transformation properties:

$$\left.\begin{array}{l} \Phi \mapsto \exp(-2iq\Lambda)\,\Phi \\ V \mapsto V \;+\; i\,\big(\Lambda \;-\; \Lambda^\dagger\big) \end{array}\right\} \quad\Rightarrow\quad \Phi^\dagger\,\exp(2qV)\,\Phi \subset K \quad \text{is gauge invariant.}$$

Here, Λ is the chiral superfield defining the generalised gauge transformations. Note that $\exp(-2iq\Lambda)\Phi$ is also chiral if Φ is.
Before supersymmetry, we defined

$$F_{\mu\nu} \;=\; \partial_\mu V_\nu \;-\; \partial_\nu V_\mu$$

as an abelian field - strength. The supersymmetric analogy is

$$W_\alpha \;:=\; -\frac{1}{4}\,(\bar{\mathcal{D}}\bar{\mathcal{D}})\,\mathcal{D}_\alpha V$$

which is chiral.

Question: How does one know that W_α is chiral?

To obtain W_α in components, it is most convenient to rewrite V in the shifted $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ variable (where $\theta\sigma^\mu\bar{\theta}V_\mu(x) = \theta\sigma^\mu\bar{\theta}V_\mu(y) - \frac{i}{2}\theta^2\bar{\theta}^2\partial_\mu V^\mu(y)$), then the supercovariant derivatives simplify to

$\mathcal{D}_\alpha = \partial_\alpha + 2i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu$ and $\bar{\mathcal{D}}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}}$:

$$\boxed{W_\alpha(y,\theta) \;=\; \lambda_\alpha(y) \;+\; \theta_\alpha\, D(y) \;+\; (\sigma^{\mu\nu}\,\theta)_\alpha\, F_{\mu\nu}(y) \;-\; i(\theta\theta)\,(\sigma^\mu)_{\alpha\dot{\beta}}\,\partial_\mu\bar{\lambda}^{\dot{\beta}}(y)}$$

Hence, we see generalised gauge invariance of W_α: λ, D and $F_{\mu\nu}$ are all gauge invariant!

4.2.4 Non - abelian field strength: non-examinable

In this section supersymmetric $U(1)$ gauge theories are generalised to non-abelian gauge groups. The gauge degrees of freedom then take values in the associated Lie algebra spanned by hermitian generators T^a:

$$\Lambda \;=\; \Lambda_a T^a\,, \qquad V \;=\; V_a T^a\,, \qquad \Big[T^a\,,\,T^b\Big] \;=\; if^{abc}\,T_c$$

Just like in the abelian case, we want to keep $\Phi^\dagger e^{2qV}\Phi$ invariant under the gauge transformation $\Phi \mapsto e^{iq\Lambda}\Phi$, but the non-commutative nature of Λ and V enforces a nonlinear transformation law $V \mapsto V'$:

$$\begin{aligned} \exp(2qV') &= \exp(iq\Lambda^\dagger)\,\exp(2qV)\,\exp(-iq\Lambda) \\ \Rightarrow \quad V' &= V \;-\; \frac{i}{2}\,(\Lambda - \Lambda^\dagger) \;-\; \frac{iq}{2}\,\Big[V\,,\,\Lambda + \Lambda^\dagger\Big] \;+\; ... \end{aligned}$$

The commutator terms are due to the Baker Campbell Hausdorff formula for matrix exponentials

$$\exp(X)\,\exp(Y) \;=\; \exp\bigg(X \;+\; Y \;+\; \frac{1}{2}\,\Big[X\,,\,Y\Big] \;+\; ...\bigg)\,.$$

The field strength superfield W_α also needs some modification in non-abelian theories. Recall that the field strength tensor $F_{\mu\nu}$ of non-supersymmetric Yang Mills theories transforms to $UF_{\mu\nu}U^{-1}$ under unitary transformations. Similarly, we define

$$W_\alpha \;:=\; -\frac{1}{8\,q}\,(\bar{\mathcal{D}}\bar{\mathcal{D}})\,\big(\exp(-2qV)\,\mathcal{D}_\alpha\,\exp(2qV)\big)$$

and obtain a gauge covariant quantity.

In Wess Zumino gauge, the supersymmetric field strength can be evaluated as

$$\boxed{\begin{aligned} W^a_\alpha(y,\theta) &= -\frac{1}{4}\,(\bar{\mathcal{D}}\bar{\mathcal{D}})\,\mathcal{D}_\alpha\,\big(V^a(y,\theta,\bar{\theta}) \;+\; i\,V^b(y,\theta,\bar{\theta})\,V^c(y,\theta,\bar{\theta})\,f^a{}_{bc}\big) \\ &= \lambda^a_\alpha(y) \;+\; \theta_\alpha\,D^a(y) \;+\; (\sigma^{\mu\nu}\,\theta)_\alpha\,F^a_{\mu\nu}(y) \;-\; i(\theta\theta)\,(\sigma^\mu)_{\alpha\dot{\beta}}\,D_\mu\bar{\lambda}^{a\dot{\beta}}(y) \end{aligned}}$$

where

$$\begin{aligned} F^a_{\mu\nu} &:= \partial_\mu V^a_\nu \;-\; \partial_\nu V^a_\mu \;+\; q\,f^a{}_{bc}\,V^b_\mu\,V^c_\nu \\ D_\mu\bar{\lambda}^a &:= \partial_\mu\bar{\lambda}^a \;+\; q\,V^b_\mu\,\bar{\lambda}^c\,f_{bc}{}^a \end{aligned}$$

4.2.5 Abelian vector superfield Lagrangian

Before attacking vector superfield Lagrangians, let us first discuss how we ensured gauge invariance of $\partial^\mu\varphi\partial_\mu\varphi^*$ under local transformations $\varphi \mapsto \exp\big(iq\alpha(x)\big)$ in the non-supersymmetric case.

- Introduce covariant derivative D_μ depending on gauge potential A_μ

$$D_\mu\varphi \;\; := \;\; \partial_\mu\varphi \; - \; iq\,A_\mu\,\varphi \; , \qquad A_\mu \;\; \mapsto \;\; A_\mu \; + \; \partial_\mu\alpha$$

  and rewrite kinetic term as

$$\mathcal{L} \;\; = \;\; D^\mu\varphi\,(D_\mu\varphi)^* + ...$$

- Add a kinetic term for A_μ to $\mathcal{L}$

$$\mathcal{L} \;\; = \;\; ... \; + \; \frac{1}{4g^2}\,F_{\mu\nu}\,F^{\mu\nu} \; , \qquad F_{\mu\nu} \;\; = \;\; \partial_\mu A_\nu \; - \; \partial_\nu A_\mu \; .$$

With SUSY, the Kähler potential $K = \Phi^\dagger\Phi$ is not invariant under

$$\Phi \;\; \mapsto \;\; \exp(-2iq\Lambda)\,\Phi \; , \qquad \Phi^\dagger\,\Phi \;\; \mapsto \;\; \Phi^\dagger\,\exp\big(-2iq(\Lambda\,-\,\Lambda^\dagger)\big)\,\Phi$$

for chiral Λ. Our procedure to construct a suitable Lagrangian is analogous to the non-supersymmetric case (although the expressions look slightly different):

- Introduce V such that

$$K \;\; = \;\; \Phi^\dagger\,\exp(2qV)\,\Phi \; , \qquad V \;\; \mapsto \;\; V \; + i\,\big(\Lambda \; - \; \Lambda^\dagger\big) \; ,$$

  i.e. K is invariant under our generalised gauge transformation.

- Add kinetic term for V with coupling τ

$$\mathcal{L}_{kin} \;\; = \;\; f(\Phi)\,(W^\alpha\,W_\alpha)\Big|_F \; + \; h.c.$$

  which is renormalisable if $f(\Phi)$ is a constant $f = \tau$. Sometimes in this case we write $\Re(\tau) = 1/g^2$. For general $f(\Phi)$, however, it is non-renormalisable. We will call f the *gauge kinetic function*.

- A new ingredient of supersymmetric theories is that an extra term can be added to $\mathcal{L}$. It is also SUSY/gauge invariant (for $U(1)$ gauge theories) and known as the *Fayet Iliopoulos term*:

$$\mathcal{L}_{FI} \;\; = \;\; \xi\,V\Big|_D \;\; = \;\; \frac{1}{2}\,\xi\,D$$

  The parameter ξ is a constant. Notice that the FI term is gauge invariant for a $U(1)$ theory because the corresponding gauge field is not charged under $U(1)$ (the photon is chargeless), whereas for a non-abelian gauge theory the gauge fields (and their corresponding D terms) would transform under the gauge group and therefore have to be forbidden. This is the reason the FI term only exists for abelian gauge theories.

The renormalisable Lagrangian of super QED involves $f = \tau = \frac{1}{4}$:

$$\mathcal{L} \;=\; \left(\Phi^\dagger \, \exp(2qV)\,\Phi\right)\Big|_D \;+\; \left(W(\Phi)\Big|_F \;+\; h.c. \right) \;+\; \left(\frac{1}{4}\, W^\alpha \, W_\alpha \Big|_F \;+\; h.c. \right) \;+\; \xi \, V\Big|_D \, .$$

If there were only one superfield Φ charged under $U(1)$ then $W = 0$. For several superfields the superpotential W is constructed out of holomorphic combinations of the superfields which are gauge invariant. In components (using Wess Zumino gauge):

$$\begin{aligned}\left(\Phi^\dagger \, \exp(2qV)\,\Phi\right)\Big|_D \;=\;& F^* F \;+\; \partial_\mu \varphi \, \partial^\mu \varphi^* \;-\; i\bar{\psi}\,\bar{\sigma}^\mu \, \partial_\mu \psi \;+\; q\, V^\mu \left(-\bar{\psi}\,\bar{\sigma}_\mu \,\psi \;+\; i\varphi^* \, \partial_\mu \varphi \;-\; i\varphi \, \partial_\mu \varphi^* \right) \\ &+\; \sqrt{2}\, q \left(\varphi \, \bar{\lambda}\bar{\psi} \;+\; \varphi^* \, \lambda \psi \right) \;+\; q\,\left(D \;+\; q\, V_\mu \, V^\mu \right)\, |\varphi|^2\end{aligned}$$

Note that

- $V^{n \geq 3} = 0$ due to Wess Zumino gauge
- we can augment ∂_μ to $D_\mu = \partial_\mu + iqV_\mu$ by soaking up the terms $\sim qV_\mu$
- only chargeless products of Φ_i may contribute in $W(\Phi_i)$, since for example $\Phi_1 \Phi_2 \Phi_3 \to \exp(-2i\Lambda(q_1 + q_2 + q_3))\Phi_1\Phi_2\Phi_3$ under a $U(1)$ gauge transformation.

In gauge theories, we have $W(\Phi) = 0$ if there is only one Φ with a non-zero charge.
Let us examine the $W^\alpha W_\alpha$- term:

$$W^\alpha \, W_\alpha \Big|_F \;=\; D^2 \;-\; \frac{1}{2}\, F_{\mu\nu}\, F^{\mu\nu} \;-\; 2i\,\lambda\,\sigma^\mu \,\partial_\mu \bar{\lambda} \;-\; \frac{i}{4}\, F_{\mu\nu}\, \tilde{F}^{\mu\nu} \, .$$

In the QED choice $f = \frac{1}{4}$, the kinetic terms for the vector superfields are given by

$$\mathcal{L}_{kin} \;=\; \frac{1}{4}\, W^\alpha \, W_\alpha \Big|_F \;+\; h.c. \;=\; \frac{1}{2}\, D^2 \;-\; \frac{1}{4}\, F_{\mu\nu}\, F^{\mu\nu} \;-\; i\lambda\,\sigma^\mu \,\partial_\mu \bar{\lambda} \, .$$

The last term in $W^\alpha W_\alpha\big|_F$ involving $\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ drops out whenever $f(\Phi)$ is chosen to be real. Otherwise, it couples as $\frac{1}{2}\mathrm{Im}\{f(\Phi)\} F_{\mu\nu}\tilde{F}^{\mu\nu}$ where $F_{\mu\nu}\tilde{F}^{\mu\nu}$ itself is a total derivative without any local physics.
With the FI contribution $\xi\, V\big|_D = \frac{1}{2}\xi D$, the collection of the D dependent terms in $\mathcal{L}$

$$\mathcal{L}_{(D)} \;=\; q\, D\, |\varphi|^2 \;+\; \frac{1}{2}\, D^2 \;+\; \frac{1}{2}\, \xi \, D$$

yields field equations

$$\frac{\delta \mathcal{S}_{(D)}}{\delta D} \;=\; 0 \quad \Longrightarrow \quad D \;=\; -\frac{\xi}{2} \;-\; q\,|\varphi|^2 \, .$$

Substituting those back into $\mathcal{L}_{(D)}$,

$$\mathcal{L}_{(D)} \;=\; -\frac{1}{2} \left(\frac{\xi}{2} \;+\; q\,|\varphi|^2 \right)^2 \;=:\; -V_{(D)}(\varphi) \, ,$$

we get a scalar potential $V_{(D)}(\varphi)$. Together with $V_{(F)}(\varphi)$ from the previous section, the total potential is given by

$$V(\varphi) \;=\; V_{(F)}(\varphi) \;+\; V_{(D)}(\varphi) \;=\; \left|\frac{\partial W}{\partial \varphi}\right|^2 \;+\; \frac{1}{2}\left(\frac{\xi}{2} \;+ q\,|\varphi|^2\right)^2 \;\geq 0.$$

Note that one always expands fields around their VEVs. The VEVs are nearly always zero, but if the scalar potential predicts a non-zero VEV v for the real part of a complex scalar field ϕ, say, one writes: $\phi = (v + h^0 + iA^0)/\sqrt{2}$, where h^0 and A^0 are real scalar fields.

In the non-abelian extension, $\xi \to 0$ and $V_{(D)}(\varphi) := \frac{1}{2} D^a D^a$, where $D^a = \varphi_i^* T_{ij}^a \varphi_j$, where a is an adjoint group label, and i, j are elements of the representation of φ. Also, $\Lambda := \Lambda_a T^a$, $V := V_a T^a$, and there are other less trivial complications in W_α and in the generalised gauge transformations as well. See Bailin and Love for all of the details.

4.2.6 Action as a superspace integral

Without SUSY, the relationship between the action $\mathcal{S}$ and $\mathcal{L}$ is

$$\mathcal{S} \;=\; \int \mathrm{d}^4 x \; \mathcal{L} \; .$$

To write down a similar expression for SUSY - actions, recall

$$\int \mathrm{d}^2\theta \; (\theta\theta) \;=\; 1 \; , \qquad \int \mathrm{d}^4\theta \; (\theta\theta) \, (\bar{\theta}\bar{\theta}) \;=\; 1 \; .$$

This provides elegant ways of expressing $K\Big|_D$ and so on:

$$\begin{aligned} \mathcal{L} \;&=\; K\Big|_D \;+\; \left(W\Big|_F \;+\; h.c.\right) \;+\; \left(f\; W^\alpha W_\alpha\Big|_F \;+\; h.c.\right) + \xi V|_D \\ &=\; \int \left(\mathrm{d}^4\theta \; K + \xi V\right) \;+\; \left(\int \mathrm{d}^2\theta \; W \;+\; h.c.\right) \;+\; \left(\int \mathrm{d}^2\theta \; f\; W^\alpha W_\alpha \;+\; h.c.\right) \end{aligned}$$

We end up with the most general action involving several chiral superfields Φ_i

$$\boxed{\begin{aligned} \mathcal{S}\Big[K\big(\Phi_i^\dagger, \exp(2qV), \Phi_i\big), W\big(\Phi_i\big), f\big(\Phi_i\big), \xi\Big] \;&=\; \int \mathrm{d}^4 x \int \mathrm{d}^4\theta \; \big(K \;+\; \xi\, V\big) \\ &\quad+\; \int \mathrm{d}^4 x \int \mathrm{d}^2\theta \; \big(W \;+\; f\, W^\alpha W_\alpha \;+\; h.c.\big) \; . \end{aligned}}$$

Recall that the FI term ξV can only appear in abelian $U(1)$ gauge theories and that the non-abelian generalisation of the $W^\alpha W_\alpha$ term requires an extra trace to keep it gauge invariant:

$$\mathrm{Tr}\Big\{W^\alpha \, W_\alpha\Big\} \;\;\mapsto\;\; \mathrm{Tr}\Big\{e^{iq\Lambda} \, W^\alpha \, W_\alpha \, e^{-iq\Lambda}\Big\} \;\;=\;\; \mathrm{Tr}\Big\{W^\alpha \, W_\alpha \; \underbrace{e^{-iq\Lambda} \, e^{iq\Lambda}}_{=\,1}\Big\}$$

Thus, we have seen that in general the functions K, W, f and the FI constant ξ determine the full structure of $\mathcal{N} = 1$ supersymmetric theories (up to two derivatives of the fields as usual). If we know their expressions we know all the interactions among the fields.

4.3 $\mathcal{N}=2,4$ global supersymmetry

For $\mathcal{N}=1$ SUSY, we had an action $\mathcal{S}$ depending on K, W, f and ξ. What will the $\mathcal{N}\geq 2$ actions depend on?
We know that in global supersymmetry, the $\mathcal{N}=1$ actions are particular cases of non-supersymmetric actions (in which some of the couplings are related, the potential is positive, etc.). In the same way, actions for extended supersymmetries are particular cases of $\mathcal{N}=1$ supersymmetric actions and will therefore be determined by K, W, f and ξ. The extra supersymmetry will put constraints to these functions and therefore the corresponding actions will be more constrained. The larger the number of supersymmetries the more constraints on actions arise.

4.3.1 $\mathcal{N}=2$

Consider the $\mathcal{N}=2$ vector multiplet

$$\begin{array}{ccc} & A_\mu & \\ \lambda & & \psi \\ & \varphi & \end{array}$$

where the A_μ and λ are described by a vector superfield V and the φ, ψ by a chiral superfield Φ.
$\mathcal{N}=2$ SUSY enforces $W=0$ in the action. K and f can be written in terms of a single holomorphic function $\mathcal{F}(\Phi)$ called the *prepotential*:

$$f(\Phi) \ = \ \frac{\partial^2\mathcal{F}}{\partial\Phi^2}\,, \qquad K(\Phi,\Phi^\dagger) \ = \ \frac{1}{2i}\left(\Phi^\dagger \exp(2V)\,\frac{\partial\mathcal{F}}{\partial\Phi} \ - \ h.c.\right)$$

The full perturbative action does not contain any corrections for more than 1 loop,

$$\mathcal{F} \ = \ \begin{cases} \Phi^2 & : \ \text{(tree level)} \\ \Phi^2 \, \ln\left(\frac{\Phi^2}{\Lambda^2}\right) & : \ \text{(1 loop)} \end{cases}$$

where Λ denotes some cutoff. These statements apply to the Wilsonian effective action. Note that:

- Perturbative processes usually involve series $\sum_n a_n g^n$ with small coupling $g \ll 1$.
- $\exp\left(-\frac{c}{g^2}\right)$ is a non-perturbative example (no expansion in powers of g possible).

There are obviously more things in QFT than Feynman diagrams can tell, e.g. instantons and monopoles.
We decompose the $\mathcal{N}=2$ prepotential $\mathcal{F}$ as

$$\mathcal{F}(\Phi) \ = \ \mathcal{F}_{\text{1loop}} \ + \ \mathcal{F}_{\text{non-pert}}$$

where $\mathcal{F}_{\text{non-pert}}$ for instance could be the *instanton expansion* $\sum_k a_k \exp\left(-\frac{c}{g^2}k\right)$. In 1994, Seiberg and Witten achieved such an expansion in $\mathcal{N}=2$ SUSY [11].

4.3.2 $\mathcal{N} = 4$

As an $N = 4$ example, consider the vector multiplet,

$$\underbrace{\begin{pmatrix} & A_\mu & \\ \lambda & & \psi_1 \\ & \varphi_1 & \end{pmatrix}}_{\mathcal{N}=2 \text{ vector}} + \underbrace{\begin{pmatrix} & \varphi_2 & \\ \psi_3 & & \psi_2 \\ & \varphi_3 & \end{pmatrix}}_{\mathcal{N}=2 \text{ hyper}} .$$

In $\mathcal{N} = 4$, there are no free functions at all, but we have a free parameter:

$$f \;=\; \tau \;=\; \underbrace{i\frac{\Theta}{2\pi}}_{F_{\mu\nu}\tilde{F}^{\mu\nu}} \;+\; \underbrace{\frac{4\pi}{g^2}}_{F_{\mu\nu}F^{\mu\nu}}$$

$N = 4$ is a finite theory, moreover its β function vanishes. Couplings remain constant at any scale, therefore we have *conformal invariance*. There are nice transformation properties under modular S *duality*,

$$\tau \;\mapsto\; \frac{a\tau + b}{c\tau + d} ,$$

where a, b, c, d form a $SL(2, \mathbb{Z})$ matrix. Finally, as an aside, major developments in string and field theories have led to the realisation that certain theories of gravity in Anti de Sitter space are "dual" to field theories (without gravity) in one less dimension, that happen to be invariant under conformal transformations. This is the *AdS/CFT correspondence* allowing one to describe gravity (and string) theories in domains where they are not well understood (the same benefit applies to field theories as well). The prime example of this correspondence is AdS in 5 dimensions dual to a conformal field theory in 4 dimensions that happens to possess $\mathcal{N} = 4$ supersymmetry.

4.4 Non-renormalisation theorems

There are some important properties of K, W, f and ξ in $N = 1$ SUSY. It was shown by using supergraph perturbation theory (a generalisation of the usual Feynman rules to superspace), that any radiative corrections in a SUSY theory can be written as $\int d^4\theta \; g$, where the function g contains *no* δ functions of θ or $\bar{\theta}$. This result (and some other similar ones) imply that:

- The interactions in K are corrected order by order in perturbation theory
- $W(\Phi)$ and ξ are *not* renormalised in perturbation theory
- $f(\Phi)$ only receives one loop - corrections

The non-renormalisation of the superpotential is one of the most important results in supersymmetric field theory. The simple behaviour of f and the non-renormalisation of ξ also have interesting consequences.

4.4.1 History

In 1977 Grisaru, Siegel, Rocek showed using "supergraphs" that, except for 1 loop corrections in f, quantum corrections only come in the form

$$\int d^4x \int d^4\theta \left\{...\right\} .$$

In 1993, Seiberg (based on string theory arguments by Witten 1985) used symmetry and holomorphicity arguments to establish these results in a simple and elegant way [10]. For more details, see Ref. [5] (section 27.6).

4.5 A few facts about local supersymmetry

We have seen that a superfield Φ transforms under supersymmetry as

$$\delta\Phi \;=\; i\,(\epsilon\mathcal{Q} \;+\; \bar{\epsilon}\bar{\mathcal{Q}})\,\Phi \;.$$

The question arises if we can make ϵ a function of space-time coordinates $\epsilon(x)$, i.e. extend SUSY to a local symmetry. The answer is yes, and the corresponding theory is *supergravity*. How did we deal with local $\alpha(x)$ in internal symmetries? We introduced a gauge field A_μ coupling to a current J^μ via an interaction term $A_\mu J^\mu$. The current J^μ is conserved and the corresponding charge q is constant

$$q \;=\; \int d^3x \; J^0 \;=\; const \;.$$

When we make the Poincaré parameters space-time dependent, we obtain a theory of gravity. The metric $g_{\mu\nu}$ as a gauge field couples to the "current" $T^{\mu\nu}$ via $g_{\mu\nu}T^{\mu\nu}$. Conservation $\partial_\mu T^{\mu\nu} = 0$ implies constant total momentum

$$P^\mu \;=\; \int d^3x \; T^{\mu 0} \;=\; const \;.$$

Now consider local SUSY. The generalised gauge field is the spin 3/2 *gravitino* Ψ^μ_α with associated *supercurrent* $\mathcal{J}^\mu_\alpha$ and SUSY charge

$$Q_\alpha \;=\; \int d^3x \; \mathcal{J}^0_\alpha \;.$$

The scalar potential of global SUSY V_F is modified in supergravity to (where $\partial_i = \frac{\partial}{\partial\varphi_i}$):

$$\boxed{V_F \;=\; \exp\left(\frac{K}{M_{\rm pl}^2}\right)\left\{(K^{-1})^{i\bar{j}}\,D_iW\,D_{\bar{j}}W^* \;-\; 3\,\frac{|W|^2}{M_{\rm pl}^2}\right\}}$$

$$D_iW \;:=\; \partial_iW \;+\; \frac{1}{M_{\rm pl}^2}\,(\partial_iK)\,W \;.$$

Note that in the $M_{\rm pl} \to \infty$ limit, gravity is decoupled and the global supersymmetric scalar potential $V_F = (K^{-1})^{i\bar{j}}\partial_iW\partial_{\bar{j}}W^*$ restored. Notice that for finite values of the Planck mass, the potential V_F above is no longer positive. The extra (negative) factor proportional to $-3|W|^2$ comes from the auxiliary fields of the gravity multiplet.

5 Supersymmetry breaking

5.1 Preliminaries

We know that fields φ_i of gauge theories transform as

$$\varphi_i \;\;\mapsto\;\; \big(\exp(i\alpha^a T^a)\big)_i{}^j\,\varphi_j\;, \qquad \delta\varphi_i \;\;=\;\; i\alpha^a\,(T^a)_i{}^j\,\varphi_j$$

under finite and infinitesimal group elements. By Goldstone's theorem, gauge symmetry is broken[8] if the vacuum state $(\varphi_{\rm vac})_i$ transforms in a non-trivial way, i.e.

$$(\alpha^a T^a)_i{}^j\,(\varphi_{\rm vac})_j \;\;\neq\;\; 0\;.$$

$\varphi_{\rm vac}$ is the value that the field φ takes when it minimises the potential $V(\varphi)$. Suppose we have a $U(1)$ symmetry, and let $\varphi = \rho\exp(i\vartheta)$ in complex polar coordinates, then infinitesimally

$$\delta\varphi \;\;=\;\; i\alpha\,\varphi \;\;\Longrightarrow\;\; \delta\rho \;\;=\;\; 0\;, \qquad \delta\vartheta \;\;=\;\; \alpha\;.$$

θ corresponds to the massless Goldstone boson (this is eaten by the gauge boson via the Higgs mechanism if the $U(1)$ is a gauge symmetry).
Similarly, we speak of broken SUSY if the vacuum state $|{\rm vac}\rangle$ satisfies

$$Q_\alpha\,|{\rm vac}\rangle \;\;\neq\;\; 0\;.$$

Let us consider the anticommutation relation $\{Q_\alpha,\bar{Q}_{\dot\beta}\} = 2(\sigma^\mu)_{\alpha\dot\beta}P_\mu$ contracted with $(\bar\sigma^\nu)^{\dot\beta\alpha}$,

$$(\bar\sigma^\nu)^{\dot\beta\alpha}\left\{Q_\alpha\;,\;\bar{Q}_{\dot\beta}\right\} \;\;=\;\; 2\,(\bar\sigma^\nu)^{\dot\beta\alpha}\,(\sigma^\mu)_{\alpha\dot\beta}\,P_\mu \;\;=\;\; 4\,\eta^{\mu\nu}\,P_\mu \;\;=\;\; 4\,P^\nu\;,$$

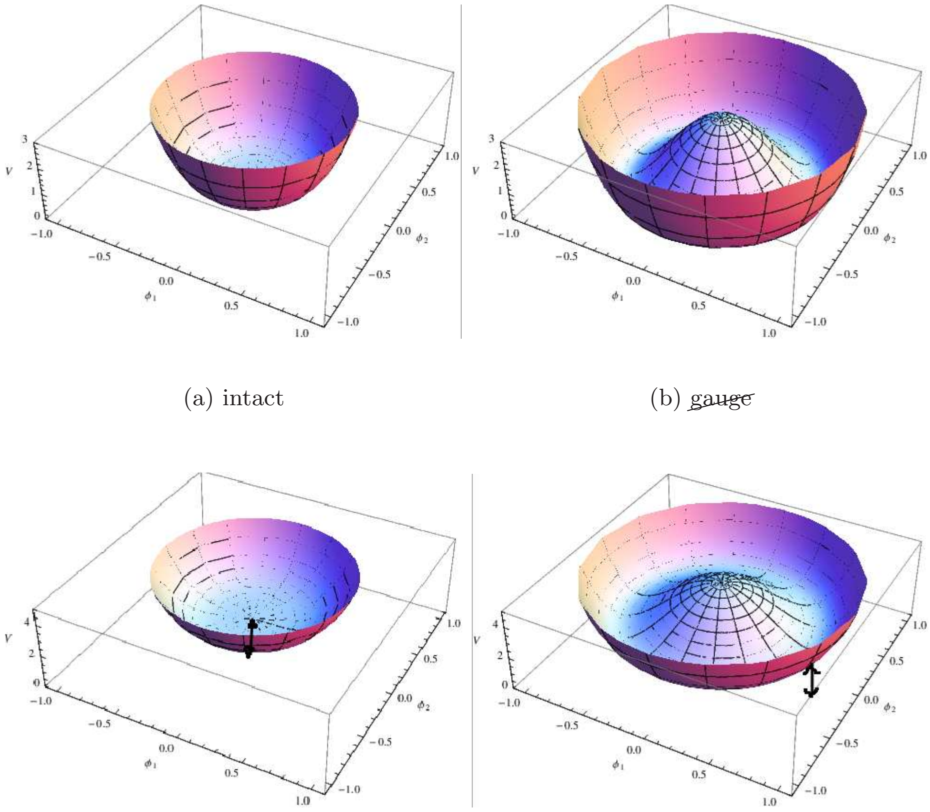
in particular the $(\nu=0)$ component using $\bar\sigma^0 = \mathbb{1}$:

$$(\bar\sigma^0)^{\dot\beta\alpha}\left\{Q_\alpha\;,\;\bar{Q}_{\dot\beta}\right\} \;\;=\;\; \sum_{\alpha=1}^{2}\big[Q_\alpha\,(Q_\alpha)^\dagger \;+\; (Q_\alpha)^\dagger\,Q_\alpha\big] \;\;=\;\; 4\,P^0 \;\;=\;\; 4\,E$$

This has two very important implications:

- $E \geq 0$ for any state, since $Q_\alpha(Q_\alpha)^\dagger + (Q_\alpha)^\dagger Q_\alpha$ is positive semi-definite
- In broken SUSY, $Q_\alpha\,|{\rm vac}\rangle \;\neq\; 0$, so $\langle{\rm vac}|[Q_\alpha(Q_\alpha)^\dagger + (Q_\alpha)^\dagger Q_\alpha]|{\rm vac}\rangle > 0$, hence the energy density is strictly positive, $E > 0$

Since W is not renormalised to all orders in perturbation theory, we have an important result: If global SUSY is unbroken at tree level, then it also unbroken to all orders in perturbation theory. This means that in order to break supersymmetry spontaneously, one has to do it non-perturbatively.



(a) intact (b) ~~gauge~~

(c) ~~SUSY~~ (d) ~~SUSY~~ ~~gauge~~

Figure 5. Various symmetry breaking scenarios: SUSY is broken, whenever the minimum potential energy $V(\varphi_{\min})$ is nonzero. Gauge symmetry is broken whenever the potential's minimum is attained at a nonzero field configuration $\varphi_{\min} \neq 0$ of a gauge non-singlet.

5.1.1 F term breaking

Consider the transformation - laws under SUSY for components of a chiral superfield Φ,

$$\begin{aligned} \delta\varphi &= \sqrt{2}\,\epsilon\psi \\ \delta\psi &= \sqrt{2}\,\epsilon\,F \;+\; i\sqrt{2}\,\sigma^\mu\,\bar{\epsilon}\,\partial_\mu\varphi \\ \delta F &= i\sqrt{2}\,\bar{\epsilon}\,\bar{\sigma}^\mu\,\partial_\mu\psi\ . \end{aligned}$$

If one of $\langle\delta\varphi\rangle$, $\langle\delta\psi\rangle$, $\langle\delta F\rangle \neq 0$, then SUSY is broken. But to preserve Lorentz invariance, we need

$$\langle\psi\rangle \;=\; \langle\partial_\mu\varphi\rangle \;=\; 0$$

as they both transform non-trivially under the Lorentz group. So our SUSY breaking condition simplifies to

$$\cancel{\text{SUSY}} \iff \langle F\rangle \neq 0\ .$$

[8]See spontaneous symmetry breaking notes in the Standard Model course.

Only the fermionic part of Φ will change,

$$\langle \delta \varphi \rangle \;=\; \langle \delta F \rangle \;=\; 0 \;, \qquad \langle \delta \psi \rangle \;=\; \sqrt{2}\, \epsilon \, \langle F \rangle \;\neq\; 0 \;,$$

so call ψ a *Goldstone fermion* or the *goldstino* (although it is not the SUSY partner of some Goldstone boson). Remember that the F term of the global SUSY scalar potential is given by

$$V_{(F)} \;=\; K^{-1}_{i\bar{j}} \, \frac{\partial W}{\partial \varphi_i} \, \frac{\partial W^*}{\partial \varphi^*_{\bar{j}}} \;,$$

and $F-$term SUSY breaking is equivalent to a positive vacuum expectation value

$$F - \text{term } \cancel{\text{SUSY}} \;\Longleftrightarrow\; \langle V_{(F)} \rangle \;>\; 0 \;.$$

5.1.2 O'Raifertaigh model

The *O'Raifertaigh model* involves a triplet of chiral superfields Φ_1, Φ_2, Φ_3 for which the Kähler potential and superpotential are given by

$$K \;=\; \Phi_i^\dagger \, \Phi_i \;, \qquad W \;=\; g\, \Phi_1 \, (\Phi_3^2 \;-\; m^2) \;+\; M \, \Phi_2 \, \Phi_3 \;, \qquad M \;\gg\; m \;.$$

From the F field equations,

$$\begin{aligned} -F_1^* \;&=\; \frac{\partial W}{\partial \varphi_1} \;=\; g\, (\varphi_3^2 \;-\; m^2) \\ -F_2^* \;&=\; \frac{\partial W}{\partial \varphi_2} \;=\; M \, \varphi_3 \\ -F_3^* \;&=\; \frac{\partial W}{\partial \varphi_3} \;=\; 2\, g\, \varphi_1 \, \varphi_3 \;+\; M \, \varphi_2 \;. \end{aligned}$$

We cannot have $F_i^* = 0$ for all $i = 1, 2, 3$ simultaneously, so this form of W indeed breaks SUSY. In order to see some effects of the SUSY breaking, we determine the spectrum. For this, we need to minimise the scalar potential:

$$V \;=\; \left(\frac{\partial W}{\partial \varphi_i} \right) \left(\frac{\partial W}{\partial \varphi_j} \right)^* \;=\; g^2 \, |\varphi_3^2 \;-\; m^2|^2 \;+\; M^2 \, |\varphi_3|^2 \;+\; \big|2\, g\, \varphi_1 \, \varphi_3 \;+\; M \, \varphi_2\big|^2$$

If $m^2 < \frac{M^2}{2g^2}$, then the minimum of the potential is at

$$\langle \varphi_2 \rangle \;=\; \langle \varphi_3 \rangle \;=\; 0 \;, \qquad \langle \varphi_1 \rangle \text{ arbitrary} \;\Longrightarrow\; \langle V \rangle \;=\; g^2 \, m^4 \;>\; 0 \;.$$

As usual, we expand the fields around the *vacuum expectation values* $\langle \varphi_{1,2,3} \rangle$. For simplicity, we take the example of $\langle \varphi_1 \rangle = 0$ and compute the spectrum of fermions and scalars. Consider the fermion mass term

$$-\frac{1}{2} \psi_i \left\langle \frac{\partial^2 W}{\partial \varphi_i \partial \varphi_j} \right\rangle \psi_j \;=\; -\frac{1}{2} \begin{pmatrix} \psi_1 & \psi_2 & \psi_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & M \\ 0 & M & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

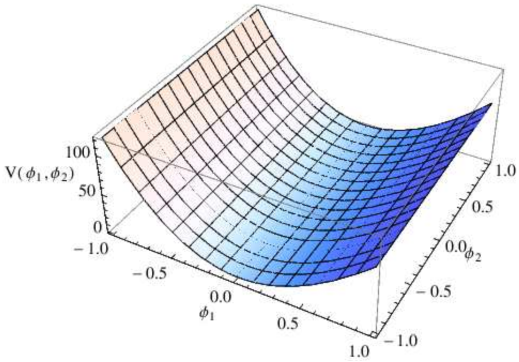


Figure 6. Example of a flat direction: If the potential takes its minimum for a continuous range of field configurations (here: for any $\varphi_2 \in \mathbb{R}$), then it is said to have a flat direction. As a result, the scalar field φ_1 will be massless.

in the Lagrangian, which yields the ψ_i masses

$$m_{\psi_1} \;=\; 0\,, \qquad m_{\psi_2} \;=\; m_{\psi_3} \;=\; M\,.$$

ψ_1 turns out to be the goldstino (due to $\langle \delta\psi_1 \rangle \propto \langle F_1 \rangle \neq 0$). To determine the scalar masses, we examine the quadratic terms in V:

$$V_{\text{quad}} \;=\; -m^2\, g^2\, (\varphi_3^2 + \varphi_3^{*2}) \,+\, M^2\, |\varphi_3|^2 \,+\, M^2\, |\varphi_2|^2 \quad\Longrightarrow\quad m_{\varphi_1} \;=\; 0\,, \qquad m_{\varphi_2} \;=\; M$$

φ_3 is a complex field, which we must split into its real and imaginary parts $\varphi_3 = \frac{1}{\sqrt{2}}(a+ib)$, since they have different masses:

$$m_a^2 \;=\; M^2 \,-\, 2\,g^2\, m^2\,, \qquad m_b^2 \;=\; M^2 \,+\, 2\,g^2\, m^2\,.$$

Summarising, we have the spectrum: show in Fig. 7.
We generally get heavier and lighter superpartners since the *supertrace* of M i.e. $\mathrm{STr}\big\{M^2\big\}$ (which treats bosonic and fermionic parts differently) vanishes:

$$\mathrm{STr}\big\{M^2\big\} \;:=\; \sum_j (-1)^{2j+1}\,(2j+1)\, m_j^2 \;=\; 0\,,$$

where j represents the 'spin' of the particles. This is generic for tree level directly broken SUSY.

5.1.3 D term breaking

Consider a vector superfield $V = (\lambda,\ V_\mu,\ D)$,

$$\delta\lambda \;\propto\; \epsilon\, D \quad\Longrightarrow\quad \langle D \rangle \;\neq\; 0 \quad\Longrightarrow\quad \cancel{\text{SUSY}}\,.$$

λ is a goldstino (which, again, is not the fermionic partner of any Goldstone boson). See examples sheet 3, where you are asked to work out some details.

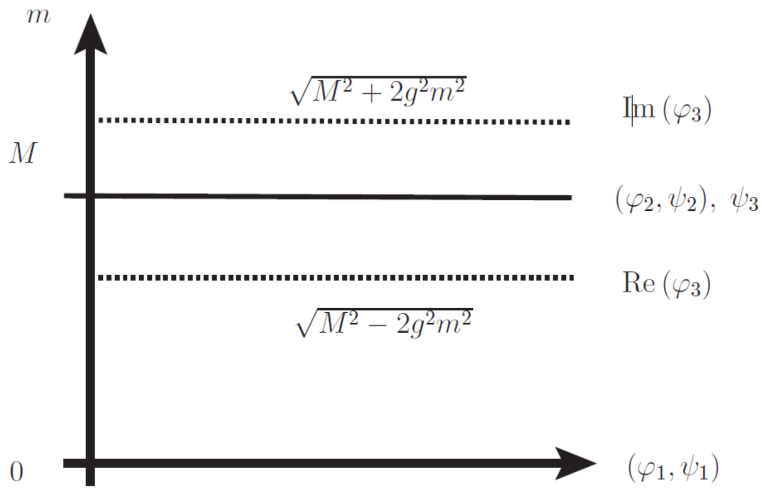


Figure 7. Mass splitting of the real- and imaginary part of the third scalar φ_3 in the O'Raifertaigh model.

5.1.4 Breaking local supersymmetry

- The supergravity multiplet contains new auxiliary - fields F_g with $\langle F_g \rangle \neq 0$ for broken SUSY.
- The F - term is proportional to

$$F \;\propto\; DW \;=\; \frac{\partial W}{\partial \varphi} \;+\; \frac{1}{M_{\mathrm{pl}}^2}\,\frac{\partial K}{\partial \varphi}\,W \;.$$

- The scalar potential $V_{(F)}$ has a negative gravitational term,

$$V_{(F)} \;=\; \exp\left(\frac{K}{M_{\mathrm{pl}}^2}\right)\left\{(K^{-1})^{i\bar{j}}\,D_iW\,D_{\bar{j}}W^* \;-\; 3\,\frac{|W|^2}{M_{\mathrm{pl}}^2}\right\} \;.$$

  That is why both $\langle V \rangle = 0$ and $\langle V \rangle \neq 0$ are possible after SUSY breaking in supergravity, whereas broken SUSY in the global case required $\langle V \rangle > 0$. This is very important for the cosmological constant problem (which is the lack of understanding of why the vacuum energy density today is almost zero, $\sim \mathcal{O}(10^{-3}\ \mathrm{eV})^4$. The vacuum energy density essentially corresponds to the vacuum expectation value of the scalar potential at its minimum. In global supersymmetry, we need to make super-particles heavy, of order ~ 100 GeV or heavier. Thus, global SUSY would naturally give a contribution to the cosmological constant that is far too large, $\sim \mathcal{O}(100\ \mathrm{GeV})^4$, since the SUSY breaking scale squared appears in the potential with no negative terms. In supergravity however, it is possible to break supersymmetry at an empirically viable large energy scale and still to keep the vacuum energy zero. This does not *solve* the cosmological constant problem, though.

- *The super Higgs effect:* Spontaneously broken gauge theories realise the Higgs mechanism in which the corresponding Goldstone boson is "eaten" by the corresponding gauge field to get a mass. A similar phenomenon happens in supersymmetry. The goldstino field joins the originally massless gravitino field (which is the gauge field of $\mathcal{N}=1$ supergravity) and gives it a mass, in this sense the gravitino receives its mass by "eating" the goldstino. The graviton remains massless, however.

6 Introducing the minimal supersymmetric standard model (MSSM)

The MSSM is based on $SU(3)_C \times SU(2)_L \times U(1)_Y \times N=1$ SUSY. We must fit all of the experimentally discovered field states into $N=1$ supermultiplets.

6.1 Particles

First of all, we have vector superfields containing the Standard Model gauge bosons. We write their representations under $(SU(3)_C,\ SU(2)_L,\ U(1)_Y)$ as (pre-Higgs mechanism):

- gluons/gluinos
$$G = (8, 1, 0)$$
- W bosons/winos
$$W = (1, 3, 0)$$
- B bosons/gauginos
$$B = (1, 1, 0),$$
which contains the gauge boson of $U(1)_Y$.

Secondly, there are chiral superfields containing Standard Model matter and Higgs fields. Since chiral superfields only contain left-handed fermions, we place charge conjugated, i.e. *anti* right handed fermionic fields (which are actually left-handed), denoted by ^{c}

- (s)quarks: lepton number $L=0$, whereas baryon number $B=1/3$ for a (s)quark, $B=-1/3$ for an anti-quark.
$$\underbrace{Q_i \;=\; \left(3,\ 2,\ \tfrac{1}{6}\right)}_{\text{left-handed}}, \qquad \underbrace{u_i^c \;=\; \left(\bar{3},\ 1,\ -\tfrac{2}{3}\right)\ , \qquad d_i^c \;=\; \left(\bar{3},\ 1,\ \tfrac{1}{3}\right)}_{\text{anti (right-handed)}}$$
- (s)leptons $L=1$ for a lepton, $L=-1$ for an anti-lepton. $B=0$.
$$\underbrace{L_i \;=\; \left(1,\ 2,\ -\tfrac{1}{2}\right)}_{\text{left-handed}}, \qquad \underbrace{e_i^c \;=\; (1,\ 1,\ +1)}_{\text{anti (right-handed)}}$$
- higgs bosons/higgsinos: $B=L=0$.
$$H_2 \;=\; \left(1,\ 2,\ \tfrac{1}{2}\right)\ , \qquad H_1 \;=\; \left(1,\ 2,\ -\tfrac{1}{2}\right)$$
the second of which is a new Higgs doublet not present in the Standard Model. Thus, the MSSM is a *two Higgs doublet model.* The extra Higgs doublet is needed in order to avoid a gauge anomaly, and to give masses to down-type quarks and leptons.

Note that after the breaking of electroweak symmetry (see the Standard Model course), the electric charge generator is $Q = T_3^{SU(2)_L} + Y/2$. Baryon and lepton number correspond to multiplicative discrete perturbative symmetries in the SM, and are thus conserved, perturbatively.
Chiral fermions may generate an *anomaly* in the theory, as shown by Fig. 8. This is where a symmetry that is present in the tree-level Lagrangian is broken by quantum corrections. Here, the symmetry is $U(1)_Y$: all chiral fermions in the theory travel in the loop, and yield a logarithmic divergence proportional to

$$A := \sum_{LH\ f_i} Y_i^3 - \sum_{RH\ f_i} Y_i^3$$

multiplied by some kinematic factor which is the same for each fermion. If A is non-zero, one must renormalise the diagram away by adding a $B_\mu B_\nu B_\rho$ counter term in the Lagrangian. But this breaks $U(1)_Y$, meaning that $U(1)_Y$ would not be a consistent symmetry at the quantum level. Fortunately, $A = 0$ for each fermion family in the Standard Model.

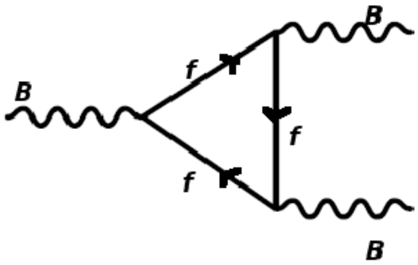


Figure 8. Anomalous graph proportional to $\mathrm{Tr}\{Y^3\}$ which must vanish for $U(1)_Y$ to be a valid symmetry at the quantum level. Hyper-charged chiral fermions f travel in the loop contributing to a three-hypercharge gauge boson B vertex.

Question: Can you show that $A = 0$ in a Standard Model family?

In SUSY, we add the Higgsino doublet $\tilde{H}_1$, which yields a non-zero contribution to A. This must be cancelled by another Higgsino doublet with opposite Y: H_2.

6.2 Interactions

- $K = \Phi_i^\dagger \exp(2V)\Phi_i$ is renormalisable, where

$$V := g_3 T^a G^a + g_2 \frac{1}{2}\sigma^i W^i + g_Y \frac{Y}{2} B,$$

  T^a being the Gell-Mann matrices and σ^i being the Pauli matrices.

- $f_a = \tau_a$ where $\mathrm{Re}\{\tau_a\} = \frac{4\pi}{g_a^2}$ determines the gauge coupling constants.

- Gauge couplings are renormalised, which ends up giving them *renormalisation scale dependence*, which matches onto dependence upon the energy scale at which one is

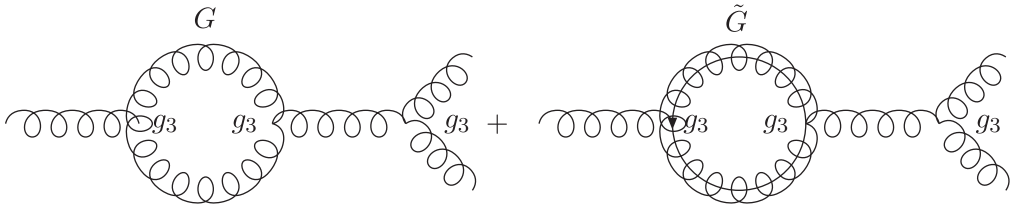


Figure 9. Contribution to the one loop QCD beta function β_3 from gluon G loops and gluino $\tilde{G}$ loops. There are other contributing diagrams, some involving loops of quarks and squarks, for instance.

probing them:

$$\mu \frac{dg_a(\mu)}{d\mu} = \beta_a g_a^3(\mu), \Rightarrow g_a^{-2}(\mu) = g_a^{-2}(\mu_0) - 2\beta_a \ln \frac{\mu}{\mu_0} \tag{6.1}$$

where β_a is a constant determined by which particles travel in the loop in the theory. For ordinary QCD it is $\beta_3 = -7/(16\pi^2)$ whereas for the MSSM, it is $\beta_3 = -3/(16\pi^2)$ because of additional contributions from squarks and gluinos to the loops, as in Fig. 9.

Eq. 6.1 is used to extrapolate gauge couplings measured at some energy scale μ_0 (often taken to be M_Z, from LEP constraints) to some other scale μ. With the SUSY contributions in the MSSM, the gauge couplings all meet at a renormalisation scale $E \approx 2 \times 10^{16}$ GeV, whereas with just the Standard Model contributions, they do not meet each other at all: see Fig. 10. The meeting of the gauge couplings is a necessary condition for a Grand Unified Theory, which only has one gauge coupling (above $M_{GUT} \approx 2 \times 10^{16}$ GeV).

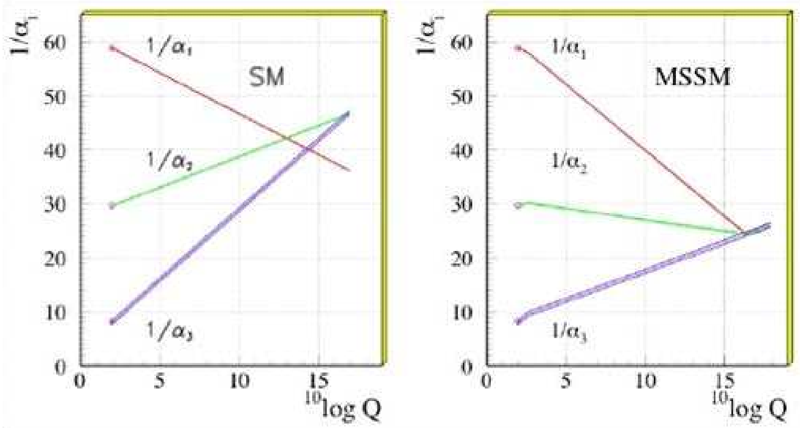


Figure 10. Renormalisation of the structure constants $\alpha_a := g_a^2/4\pi$ associated with the $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ groups.

- For the FI term: we must have $\xi = 0$, otherwise the scalar potential breaks charge and colour (because one generates a non-zero vacuum expectation value for a squark, for instance).

- We write down a superpotential containing all terms which are renormalisable and consistent with our symmetries. If one does this, one obtains two classes of terms, $W = W_{R_p} + W_{RPV}$. The terms in W_{R_p} all conserve baryon number B and lepton number L, whereas those in W_{RPV} break either B or L:

$$\begin{aligned} W_{R_p} &= (Y_U)_{ij}\, Q_i\, H_2\, u^c_j \;+\; (Y_D)_{ij}\, Q_i\, H_1\, d^c_j \;+\; Y_E\, L_i\, H_1\, e^c_j \;+\; \mu\, H_1\, H_2 \\ W_{RPV} &= \lambda_{ijk}\, L_i\, L_j\, e^c_k \;+\; \lambda'_{ijk}\, L_i\, Q_j\, d^c_k \;+\; \lambda''_{ijk}\, u^c_i\, d^c_j\, d^c_k \;+\; \kappa_i\, L_i\, H_2, \end{aligned}$$

where we have suppressed gauge indices.

Question: Which terms break L and which break B? Why is there no term $\lambda'''_k H_1 H_1 e^c_k$ in W_{RPV}?

The first three terms in W_{R_p} correspond to standard Yukawa couplings and give masses to up quarks, down quarks and leptons, as we shall see. Writing $x = 1, 2, 3$ as a fundamental $SU(3)$ index, $a, b = 1, 2$ as fundamental $SU(2)$ indices, the first term in W_{R_p} becomes

$$(Y_U)_{ij} Q^{xa}_i H^b_2 u^c_{jx} \epsilon_{ab} = (Y_U)_{ij}[u^x_L H^0_2 u^c_{jx} - d^x_L H^+_2 u^c_{jx}].$$

Once the neutral Higgs component develops a vacuum expectation value, $H^0_2 := (v_2 + h^0_2)/\sqrt{2}$, the first term becomes $(Y_U)_{ij} v_2/\sqrt{2} u^x_{Li} u^c_{jx} + \ldots$, yielding a Dirac mass matrix $m_u := (Y_U)_{ij} v_2/\sqrt{2}$ for the up quarks. The down quark and lepton masses proceed in an analogous manner. The fourth term is a mass term for the two Higgs(ino) fields.

If all of the terms in W_{RPV} are present, the interaction shown in Fig. 11 would allow proton decay $p \to e^+ + \pi^0$ within seconds, whereas experiments say that it should be $> 10^{34}$ years. In order to forbid proton decay an extra symmetry should be imposed. One symmetry that works is a discrete multiplicative symmetry R *parity* defined as

$$R \;:=\; (-1)^{3(B-L)+2S} \;=\; \begin{cases} +1 \;:\; \text{Standard Model particles} \\ -1 \;:\; \text{superpartners} \end{cases}.$$

It forbids all of the terms in W_{RPV}, but there exist other examples which only ban some subset.

R parity would have important physical implications:

- The lightest superpartner (LSP) is stable.

- Cosmological constraints then say that a stable LSP must be electrically and colour-neutral (higgsino, photino, zino). It is then a good candidate for cold weakly interacting dark matter.

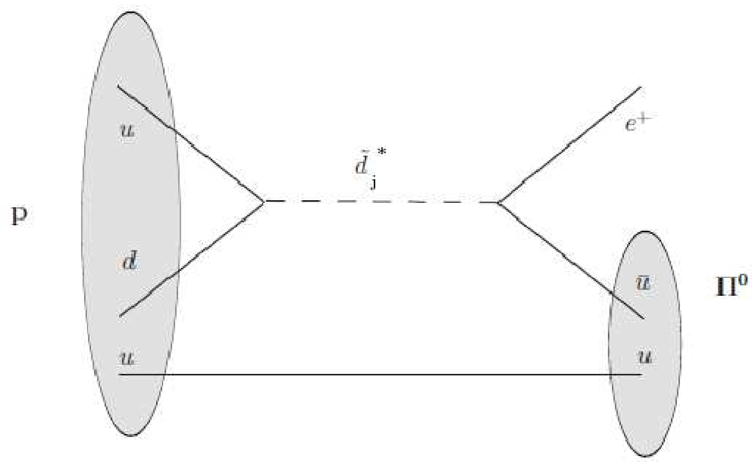


Figure 11. Proton decay due to baryon- and lepton number violating interactions. Both B and L violating terms must be present for the proton to decay. The matrix element is proportional to $\lambda''_{1j1}{}^* \times \lambda'_{11j}{}^*$.

- In colliders, the initial state is $R_p = +1$, implying that superparticles are produced in pairs. When a superparticle decays, it must decay to another (lighter) superparticle plus some standard model particles.

- One ends up with LSPs at the end of the decays. These do not interact with the detector, and hence appear as unbalanced or 'missing' momentum.

Note that the terms in W_{RPV} can lead to Majorana fermion structure[9]. For instance, $W = \lambda''_{112} u_1^c d_1^c d_2^c$: we take the $F-$ terms as usual in order to find the Lagrangian in terms of components:

$$\mathcal{L} = \frac{1}{2}\left(\lambda''_{112}\tilde{u}_1^* d_1{}^\dagger_R C d_2{}^*_R - (\lambda''_{112})^* \tilde{u}_1 d_1{}^T_R C^* d_{2R}\right)$$

plus supersymmetric copies, where C is the charge conjugation matrix and ^{T} denotes transpose.

6.3 Supersymmetry breaking in the MSSM

We cannot break supersymmetry directly in the MSSM, since it preserves $\mathrm{STr}\{M^2\} = 0$. Applying this to the photon, say: $-3m_\gamma^2 + 2m_{\tilde{\gamma}}^2 = 0$, which would predict a massless photino that hasn't been observed. Applying it to up quarks: $2m_u^2 - m_{\tilde{u}_L}^2 - m_{\tilde{u}_R}^2 = 0$, thus one up squark must be *lighter* than the up quark, again this hasn't been observed. We introduce a *hidden* sector, which breaks SUSY and has its own fields (which do not directly interact with MSSM fields) and interactions, and an additional *messenger sector*

$$\begin{pmatrix}\text{observable}\\ \text{sector, MSSM}\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{messenger -}\\ \text{sector}\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{hidden}\\ \text{sector}\end{pmatrix}.$$

[9]This is a familiar structure for people extending the Standard Model to include neutrino masses.

This gets around the supertrace rule. There is typically an overall gauge group

$$\big(SU(3) \times SU(2) \times U(1)\big) \times G_{\cancel{\rm SUSY}} =: G_{SM} \times G_{\cancel{\rm SUSY}},$$

where the MSSM fields are singlets of $G_{\cancel{\rm SUSY}}$ and the hidden sector fields are singlets of G_{SM}.

We have already seen several examples of SUSY breaking theories. One popular SUSY-breaking sector in the MSSM context is that of *gaugino condensation*: here, some asymptotically free gauge coupling g becomes large at some energy scale Λ. g will renormalise like Eq. 6.1 with some beta function coefficient. Solving the equation, with $g^{-2}(\Lambda) \to 0$, we obtain $\Lambda = M \exp[g^{-2}(M)/\beta]$. M could be some large scale such as the string scale, $\sim 5 \times 10^{17}$ GeV. It is easy to arrange for $\Lambda \ll M$. When the gauge coupling becomes large, and the theory becomes non-perturbative, one can obtain $\langle \tilde{g}\tilde{g} \rangle \sim \mathcal{O}(\Lambda^3)$, breaking SUSY dynamically[10].

The SUSY breaking fields have couplings with the messenger sector, which in turn have couplings with the MSSM fields, and carry the SUSY breaking over to them. There are several possibilities for the messenger sector fields, which may determine the explicit form of SUSY breaking terms in the MSSM, including (note here that $M_{\cancel{\rm SUSY}}$ is the SUSY breaking in the hidden sector, whereas Δm is the SUSY breaking that ends up in the MSSM fields):

- gravity mediated ~~SUSY~~

  If the mediating field couples with gravitational strength to the standard model, the couplings are suppressed by the inverse Planck mass $M_{\rm pl}$, the natural scale of gravity. The SUSY breaking mass splitting between MSSM particles and superparticles, Δm, becomes

  $$\Delta m = \frac{M^2_{\cancel{\rm SUSY}}}{M_{\rm pl}} .$$

  We want $\Delta m \approx 1$ TeV and know $M_{\rm pl} \approx 10^{18}$ GeV, so

  $$M_{\cancel{\rm SUSY}} = \sqrt{\Delta m \cdot M_{\rm pl}} \approx 10^{11} \text{ GeV} .$$

  The gravitino gets a mass $m_{\frac{3}{2}}$ of Δm order TeV from the super Higgs mechanism.

- gauge mediated ~~SUSY~~

  Messenger fields are charged under both G_{SM} and $G_{\cancel{\rm SUSY}}$. Gauge loops transmit SUSY breaking to the MSSM fields. Thus, $\Delta m \sim$ gives $M_{\cancel{\rm SUSY}}/(16\pi^2) \sim \mathcal{O}(\Delta m)$, i.e. TeV. In that case, the gravitino mass $m_{\frac{3}{2}} \sim \frac{M^2_{\cancel{\rm SUSY}}}{M_{\rm pl}} \sim$ eV and is the LSP.

- anomaly mediated ~~SUSY~~

  In this case, the auxiliary fields of supergravity get a vacuum expectation value. The effects are always present, but suppressed by loop factors. They may be dominant if the tree-level contribution is suppressed for some reason.

[10] Here, $\tilde{g}$ is the gaugino of the hidden sector gauge group, and β is the hidden gauge group beta function coefficient.

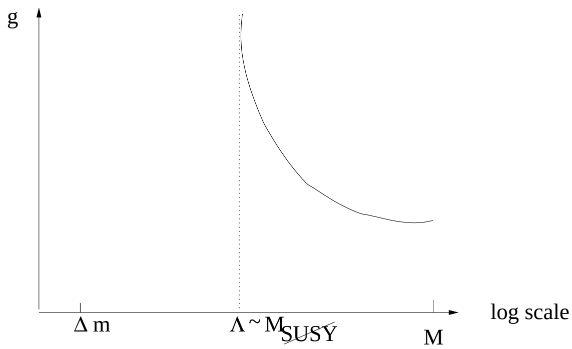


Figure 12. Gaugino condensation and supergravity mediated SUSY breaking

Each of these scenarios has phenomenological advantages and disadvantages and solving their problems is an active field of research. In all scenarios, the Lagrangian for the observable sector has contributions

$$\mathcal{L} \;=\; \mathcal{L}_{\text{SUSY}} \;+\; \mathcal{L}_{\cancel{\text{SUSY}}}.$$

In the second term, we write down all renormalisable symmetry invariant terms which do not reintroduce the hierarchy problem. They are of the form (where i and j label different fields):

$$\boxed{\mathcal{L}_{\cancel{\text{SUSY}}} \;=\; \underbrace{m_{ij}^2\,\varphi_i^*\,\varphi_j + m'^2_{ij}(\varphi_i\varphi_j + h.c.)}_{\text{scalar masses}} \;+\; \left(\underbrace{\frac{1}{2}M_\lambda\,\lambda\lambda}_{\text{gaugino masses}} \;+\; \underbrace{A_{ijk}\,\varphi_i\varphi_j\varphi_k}_{\text{trilinear couplings}} \;+\; h.c.\right).}$$

$M_\lambda, m'^2_{ij}, m^2_{ij}, A_{ijk}$ are called *soft SUSY breaking terms*: they do not reintroduce quadratic divergences into the theory. Particular forms of SUSY breaking mediation can give relations between the different soft SUSY breaking terms. They determine the amount by which supersymmetry is expected to be broken in the observable sector, and the masses of the superparticles for which the LHC is searching.

Explicitly, we parametrise all of the terms that softly break SUSY in the R_p preserving MSSM, suppressing gauge indices:

$$\begin{aligned}\mathcal{L}_{R_p}^{\cancel{\text{SUSY}}} = {} & (A_U)_{ij}\tilde{Q}_{Li}H_2\tilde{u}^*_{Rj} + (A_D)_{ij}\tilde{Q}_{Li}H_1\tilde{d}^*_{Rj} + (A_E)_{ij}\tilde{L}_{Li}H_1\tilde{e}^*_{Rj} + \\ & \tilde{Q}^*_{Li}(m^2_{\tilde{Q}})_{ij}\tilde{Q}_{Lj} + \tilde{L}^*_i(m^2_{\tilde{L}})_{ij}\tilde{L}_j + \tilde{u}_{Ri}(m^2_{\tilde{U}})_{ij}\tilde{u}^*_{Rj} + \tilde{d}_{Ri}(m^2_{\tilde{D}})_{ij}\tilde{d}^*_{Rj} + \tilde{e}_{Ri}(m^2_{\tilde{E}})_{ij}\tilde{e}^*_{Rj} + \\ & (m_3^2H_1H_2 + h.c.) + m_1^2|H_1^2| + m_2^2|H_2|^2 + \frac{1}{2}M_3\tilde{g}\tilde{g} + \frac{1}{2}M_2\tilde{W}\tilde{W} + \frac{1}{2}M_1\tilde{B}\tilde{B}.\end{aligned}$$

Sometimes, m_3^2 is written as μB. Often, specific high scale models provide relations between these many parameters. For instance, the Constrained MSSM (which may come from some

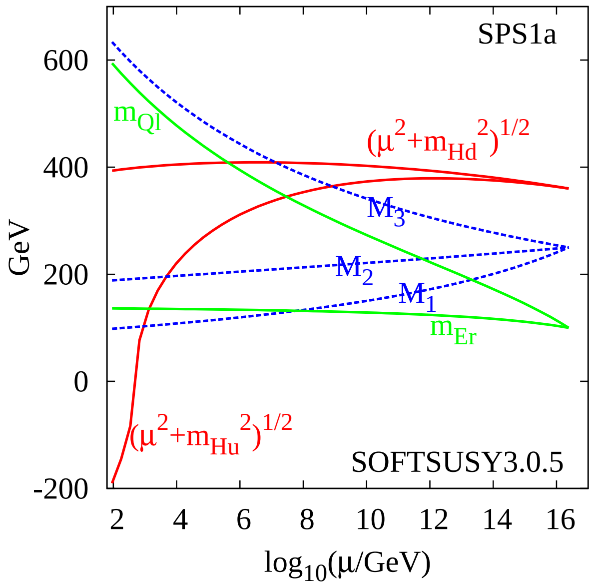


Figure 13. An example of renormalisation in the MSSM. A particular high energy theory is assumed, which has GUT symmetry and implies that the gauginos are all mass degenerate at the GUT scale. The scalars (e.g the right-handed electron Er and the left-handed squarks Ql) are also mass-degenerate at the GUT scale. Below the GUT scale though, the masses split and renormalise separately. When we are scattering at energies $\sim O(1)$ GeV, it is a good approximation to use the masses evaluated at that renormalisation scale $\mu \approx E$. We see that one of the Higgs mass squared parameters, $\mu^2 + M_{Hu}^2$, becomes negative at the electroweak scale, triggering electroweak symmetry breaking.

string theory or other field theory) gives the constraints

$$\begin{aligned} M_1 &= M_2 = M_3 =: M_{1/2} \\ m_{\tilde{Q}}^2 &= m_{\tilde{L}}^2 = m_{\tilde{U}}^2 = m_{\tilde{D}}^2 = m_{\tilde{E}}^2 := m_0^2 I_3 \\ m_1^2 &= m_2^2 = m_0^2 \\ A_U &= A_0 Y_U, \; A_D = A_0 Y_D, \; A_E = A_0 Y_E \end{aligned}$$

where I_3 is the 3 by 3 identity matrix. Thus in the 'CMSSM', we reduce the large number of free SUSY breaking parameters down to[11] 3: $M_{1/2}$, m_0 and A_0. These relations hold at the GUT scale, and receive large radiative corrections, as Fig. 13 shows.

6.4 The hierarchy problem

The Planck mass $M_{\text{pl}} \approx 10^{19}$ GeV is an energy scale associated with gravity and the electroweak scale $M_{\text{ew}} \approx 10^2$ GeV is an energy scale associated with symmetry breaking scale of the Standard Model. The hierarchy problem involves these two scales being so different in magnitude. Actually the problem can be formulated in two parts:

[11] One should really include $\tan\beta = v_2/v_1$ as well, the ratio of the two Higgs vacuum expectation values.

(i) Why is $M_{\rm ew} \ll M_{\rm pl}$ at tree level? Answering this question is the hierarchy problem. There are many solutions.

(ii) Once we have solved (i), why is this hierarchy stable under quantum corrections? This is the 'technical hierarchy problem' and does not have many solutions, aside from SUSY.

Let us now think some more about the technical hierarchy problem. In the Standard Model we know that:

- Vector bosons are massless due to gauge invariance, that means, a direct mass term for the gauge particles $M^2 A_\mu A^\mu$ is not allowed by gauge invariance ($A_\mu \to A_\mu + \partial_\mu \alpha$ for a $U(1)$ field, for example).

- Chiral fermion masses $m\psi\psi$ are also forbidden for all quarks and leptons by gauge invariance.

  Question: Which symmetry bans say $m e_R e_R$?

  Recall that these particles receive a mass only through the Yukawa couplings to the Higgs (e.g. $H\bar{\psi}_L\psi_R$ giving a Dirac mass to ψ after H gets a non-zero value[12]).

- The Higgs is the only scalar particle in the Standard Model. There is no symmetry banning its mass term $m_H^2 H^\dagger H$ in the Standard Model Lagrangian. If the heaviest state in the theory has a mass of Λ, loops give corrections of order $\Lambda^2/(16\pi^2)$ to the scalar mass. The corrections come from both bosons and fermions running in loops. Experimentally, the Higgs mass is measured to be $m_H \approx 126$ GeV. The Standard Model is considered to be unnatural since the loop corrections are typically much larger: the largest are expected to be[13] $\sim \mathcal{O}(10^{17})$ GeV. Therefore even if we start with a Higgs mass of order the electroweak scale, loop corrections would bring it up to the highest scale in the theory, $\Lambda/(16\pi^2)$. This would ruin the hierarchy between large and small scales. It is possible to adjust or "fine tune" the loop corrections such as to keep the Higgs light, but this would require cancellations between the apparently unrelated tree-level and loop contributions to some 15 significant figures. This fine tuning is considered unnatural and an explanation of why the Higgs mass (and the whole electroweak scale) can be naturally maintained to be hierarchically smaller than the Planck scale or any other large cutoff scale Λ is required.

In SUSY, bosons have the same masses as the fermions. Since quarks and leptons are massless because of gauge invariance, SUSY implies that the squarks and sleptons are protected too.

Secondly, SUSY implies that in the explicit computation of loop diagrams (see Fig. 4), the leading divergences of the bosonic loops cancel against the fermionic loops. This is due to

[12] Notice that with $R-$parity, the MSSM does not give neutrinos mass. Thus one must augment the model in some way.

[13] This does rely on quantum gravity yielding an effective quantum field theory that acts in the usual way.

the fact that the couplings defining SUSY relates the vertices in each diagram to involve the same coupling. Even when SUSY is softly broken, these leading divergences cancel, leaving us with only a term of $\mathcal{O}(\frac{1}{16\pi^2}M_{SUSY}\ln\Lambda)$, where M_{SUSY} is the SUSY breaking mass of some particle in the loop.
Therefore if supersymmetry were exact, fermions and bosons would be degenerate, but if M_{SUSY} is close to the electroweak scale then it will protect the Higgs from becoming too heavy. Thus, we expect the superparticle masses to be close to the electroweak scale, and therefore accessible at the LHC.

6.5 Pros and Cons of the MSSM

We start with a list of unattractive features of the MSSM:

- There are ~ 100 extra free parameters in the SUSY breaking sector, making for a complicated parameter space.
- Nearly all of this parameter space is ruled out from flavour physics constraints: SUSY particles could heavily mix in general, then this mixing could appear in loops and make the quarks mix in a flavour changing neutral current, upon which there are very strong experimental bounds. It could be that this clue is merely telling us that there is more structure to the MSSM parameter space, though (like in the CMSSM).
- The μ problem. μ in W_{R_p} must be $< \mathcal{O}(1)$ TeV, since it contributes at tree-level to m_H. Why should this be, when in principle we could put it to be $\sim \mathcal{O}(M_{Pl})$, because it does not break any SM symmetries? (Note though that once it is set to be small at tree-level, SUSY protects it from large quantum corrections).

These SUSY problems can be solved with further model building.

We close with an ordered list of weak-scale SUSY's successes:

- SUSY solves the technical hierarchy problem.
- Gauge unification works.
- The MSSM contains a viable dark matter candidate, if R_p is conserved.
- Electroweak symmetry breaks radiatively.

Acknowledgements

These lecture notes are heavily based on Ref. [1].

Appendix: the Part III Exam

There is a 3 hour examination for this course. You will be asked to complete 3 out of 4 possible questions. As ever, you should work through some past papers to get an idea for the kind of questions you can expect.

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