232A Lecture Notes
Representation Theory of Lorentz Group

1 Symmetries in Physics

Symmetries play crucial roles in physics. Noether's theorem relates symmetries of the system to conservation laws. In quantum mechanics, conserved quantities then become the *generators* of the symmetry. The four forces we know (gravity, strong, electromagnetic, and weak) are all based on *gauge symmetries*. Therefore, it is very important to understand how particle states and fields transform under the symmetries.

Symmetries always form mathematical *groups*. To figure out how the abstract concepts of groups transform states and fields is an issue of *representation* of the group. For representations of compact Lie groups (see below), there are many excellent textbooks. I've personally studied one by Howard Georgi [1].

On the Hilbert space of quantum systems, all symmetries must be given in terms of unitary or anti-unitary operators so that probabilities do not change before and after the symmetry transformation. If the states $|a\rangle$ and $|b\rangle$ transform to $U|a\rangle$ and $U|b\rangle$ under a unitary transformation, their inner products do not change,

$$(\langle a|U^\dagger)(U|b\rangle) = \langle a|(U^\dagger U|b\rangle) = \langle a|b\rangle. \tag{1}$$

In the case of anti-unitary operators, we find instead[1]

$$(\langle a|U^\dagger)(U|b\rangle) = [\langle a|(U^\dagger U|b\rangle)]^* = (\langle a|b\rangle)^* = \langle b|a\rangle, \tag{2}$$

which shows the initial and final states are interchanged, appropriate for time reversal. In either case, the probabilities given by the absolute square of the matrix elements remain the same.

However, the fields are not states. The quantum fields are the *canonical coordinates* in the field-theory Lagrangian, and there is no probabilistic interpretation for them. Therefore, fields may transform in non-unitary fashion under symmetries.

[1] To understand this unusual behavior of anti-unitary operators, I find the quantum mechanics textbook by Albert Messiah [2] the most useful.

In this lecture notes, I describe the representation of the Lorentz group, a group of spatial rotations and Lorentz boosts, which is hard to find in mathematics literature mostly focused on compact groups.[2] When the fields transform under the Lorentz group, we need to use non-unitary representations. On the other hand, when the states in the Hilbert space transform under the Lorentz group, we use unitary representations.

2 Lorentz Group

The Lorentz group is a symmetry of the Minkowski spacetime that leaves the origin unchanged. We adopt the metric

$$g_{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{3}$$

The proper time is defined by

$$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu = c^2(dt)^2 - (d\vec{x})^2. \tag{4}$$

(Whenever I put an arrow on top of a symbol, it refers to a three-vector in space. x, y etc without an arrow refers to the whole spacetime coordinate including time.) The Lorentz transformation is supposed to keep this combination of time and space intervals invariant.

Therefore, the Lorentz group is a collection of matrices O that have the property

$$OgO^T = g. \tag{5}$$

It consists of the regular rotations of the three-dimensional space such as

$$O = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta & \sin\theta \\ & & -\sin\theta & \cos\theta \end{pmatrix}, \tag{6}$$

[2]The original works, however, are due to mathematicians Bartel Leendert van der Waerden.

as well as invariance under Lorentz boosts such as

$$O = \begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} \cosh\eta & \sinh\eta & & \\ \sinh\eta & \cosh\eta & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{7}$$

Here, $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$, and we also introduced the rapidity parameter $\gamma = \cosh\eta$, $\gamma\beta = \sinh\eta$ using the fact that $\gamma^2 - (\gamma\beta)^2 = 1$.

The *proper Lorentz group* contains only transformations with unit determinant called $SO(3,1)$, while *improper Lorentz transformations* are those transformations with determinant -1 (either parity or time-reversal is included.) Combined, they form the group $O(3,1)$. (For these notations, see Appendix.) Therefore, a Lorentz transformation falls into one of the four distinct categories:

categories	no parity	with parity
no time-reversal	proper	improper
with time-reversal	improper	proper

3 Generators

Lorentz transformations without parity nor time reversal are generated by a set of operators (infinitesimal transformations), just like spatial rotations are generated by angular momentum operators. They satisfy commutation relations (Lie algebra). The easiest way to figure it out is to generalize the orbital angular momentum operators (in the unit of $\hbar$) in coordinate space

$$\frac{1}{\hbar}L_i = \frac{1}{\hbar}\epsilon_{ijk}x_j p_k = -i\epsilon_{ijk}x_j\nabla_k, \tag{8}$$

and define

$$M^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu). \tag{9}$$

Note that $\partial^1 = -\partial_1 = -\nabla_x$ etc because of the negative metric for spatial coordinates. Here and below, we use the notation that Greek indices are Lorentz indices $\mu = 0, 1, 2, 3$ with distinction between upper and lower indices, while the Latin indices are only space $i = x, y, z$ without distinction between upper and lower indices.

Using the fact $\partial^\mu x^\nu = g^{\mu\nu}$, it is easy to work out

$$\begin{aligned}[M^{\mu\nu}, M^{\rho\sigma}] &= [i(x^\mu\partial^\nu - x^\nu\partial^\mu), i(x^\rho\partial^\sigma - x^\sigma\partial^\rho)] \\ &= -(x^\mu g^{\nu\rho}\partial^\sigma - x^\rho g^{\mu\sigma}\partial^\nu - x^\nu g^{\mu\rho}\partial^\sigma + x^\rho g^{\nu\sigma}\partial^\mu \\ &\quad + x^\nu g^{\mu\sigma}\partial^\rho - x^\sigma g^{\nu\rho}\partial^\mu - x^\mu g^{\nu\sigma}\partial^\rho + x^\sigma g^{\mu\rho}\partial^\nu) \\ &= i(g^{\nu\rho}M^{\mu\sigma} + g^{\mu\sigma}M^{\nu\rho} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho}). \qquad (10)\end{aligned}$$

The generators of rotation are given by $J_x = M^{23}$, $J_y = M^{31}$, $J_z = M^{12}$. To see this, we can verify, *e.g.*,

$$[J_x, J_y] = [M^{23}, M^{31}] = ig^{33}M^{21} = iJ_z. \qquad (11)$$

In general,

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \qquad (12)$$

The operators $K_i = M^{0i}$ generate boost along the axis $i = x, y, z$. They satisfy the commutation relations

$$[J_x, K_y] = [M^{23}, M^{02}] = ig^{22}M^{30} = iK_z. \qquad (13)$$

In general,

$$[J_i, K_j] = i\epsilon_{ijk}K_k. \qquad (14)$$

It basically shows that boost generators rotate among themselves as a three-vector. Finally, the commutation relations among the boost operators are

$$[K_x, K_y] = [M^{01}, M^{02}] = i(-g^{00}M^{12}) = -iJ_z. \qquad (15)$$

In general,

$$[K_i, K_j] = -i\epsilon_{ijk}J_k. \qquad (16)$$

One way to verify that M^{0i} above indeed generates Lorentz boost, see

$$M^{01}t = i(x^0\partial^1 - x^1\partial^0)t = i(-t\nabla_z - x\nabla_t)t = -ix, \qquad (17)$$

$$M^{01}x = i(x^0\partial^1 - x^1\partial^0)x = i(-t\nabla_z - x\nabla_t)x = -it. \qquad (18)$$

Therefore, I can write

$$M^{01}\begin{pmatrix} t \\ x \end{pmatrix} = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix} = -i\sigma_x\begin{pmatrix} t \\ x \end{pmatrix}. \qquad (19)$$

Then it is easy to compute the exponential of this generator,

$$e^{i\eta M^{01}} = e^{\eta\sigma_x} = \sum_{n=0}^{\infty} \frac{\eta^n}{n!}(\sigma_x)^n. \tag{20}$$

Since $(\sigma_x)^2 = I_2$ (two-by-two identity matrix), all the odd powers are σ_x and even powers I_2. Therefore,

$$\begin{aligned} e^{i\eta M^{01}} \begin{pmatrix} t \\ x \end{pmatrix} &= \left[\sum_{n=\text{even}}^{\infty} \frac{\eta^n}{n!} I_2 + \sum_{n=\text{odd}}^{\infty} \frac{\eta^n}{n!}\sigma_x\right] \begin{pmatrix} t \\ x \end{pmatrix} \\ &= [\cosh\eta I_2 + \sinh\eta\ \sigma_x] \begin{pmatrix} t \\ x \end{pmatrix} \\ &= \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}. \end{aligned} \tag{21}$$

This is nothing but the boost Eq. (7). Note that this matrix is not unitary.

At this point, $M^{\mu\nu}$ (and hence $\vec{J}$ and $\vec{K}$) are abstract operators, just like $\vec{x}$, $\vec{p}$, $\vec{L} = \vec{x} \times \vec{p}$ in quantum mechanics. The task ahead of us is to come up with something more concrete, just like states $|jm\rangle$ in quantum mechanics and $J_\pm$, J_z acting on them. Namely, we need a column vector on which operators act in forms of matrices. We then have to make sure that these matrices satisfy the same commutation relations. The combination of matrices and column vectors is called a *representation* of the Lie algebra (commutation relations).

Note that the Lorentz group is *non-compact*. Imagine doing many same small transformations successively. While the rotations eventually bring back to where you started once the angle is 2π (*compact*), boost can go on for ever and does not come back because η above can go all the way to infinity (*non-compact*). It is a mathematical theorem that says that non-compact groups do not have finite-dimensional unitary representations. The reason is very simple. A unitary matrix, multiplied many many times, will eventually come back to unity. Unless the space is infinite-dimensional, so that you may never come back to where you started. Indeed, the boost Eq. (7) in a four-dimensional representation is not a unitary matrix.

For fields, we need to write their Lagrangians and do not want them to have an infinite number of components. We assign fields to transform under finite-dimensional representations of the Lorentz group. Therefore, the boost generators are represented by non-unitary matrices, as you will see explicitly later. Correspondingly, the boost generators are not hermitian.

On Hilbert space, however, we want all generators to be quantum mechanical observables and hence hermitian. Therefore, the Hilbert space must be infinite-dimensional unitary representations. We will also see later how to construct such representations.

4 Representation Theory of Lorentz Group

As mentioned earlier, we need to come up with finite-dimensional non-unitary representations of the Lorentz group to be assigned for fields (not Hilbert space). Here is how we can classify all such representations.

Using the commutation relations we derived, we can identify two sets of mutually commuting operators. Let us define

$$L_i^\pm = (J_i \pm iK_i)/2. \tag{22}$$

Then we find

$$[L_i^\pm, L_j^\pm] = [J_i \pm iK_i, J_j \pm iK_j]/4 = i\epsilon_{ijk}(J_k \pm iK_k \pm iK_k - J_k)/4 = i\epsilon_{ijk}L_k^\pm, \tag{23}$$

while the different sets commute

$$[L_i^+, L_j^-] = [J_i + iK_i, J_j - iK_j]/4 = i\epsilon_{ijk}(J_k - iK_k + iK_k - J_k)/4 = 0. \tag{24}$$

Therefore there are two sets of Lie algebras which have the same form as those of angular momentum operators $\mathfrak{su}(2)$. Therefore any finite-dimensional representations of Lorentz group are specified by two numbers, (j_+, j_-), one for the generators L_i^+ and other for L_i^-.[3] The representation is $(2j_+ +1)(2j_- +1)$ dimensional.

Note that under the rotation group, we discard the boost operators $\vec{K}$ and there is no distinction between the two sets. Therefore, both j_+ and j_- come in for spatial rotations, and we *add* angular momenta j_+ and j_- to find angular momenta $j = |j_+ - j_-|, |j_+ - j_-| + 1, \cdots j_+ + j_-$.

In order to obtain spin 1/2 under spatial rotations without anything else we don't want, we find only two inequivalent possibilities, $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. Both of them are indeed $j = 1/2$ under rotations. The difference is in the properties under the boost. Let us see how each of them transforms.

[3] Mathematically, there is an isomorphism $\mathfrak{so}(3,1) \otimes \mathbb{C} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$, where each $\mathfrak{sl}(2,\mathbb{C})$ factor picks $j_\pm$ representation.

4.1 $(\frac{1}{2}, 0)$ spinors

For the $(\frac{1}{2}, 0)$ representation, $j_- = 0$ and therefore $\vec{L}^- = \vec{J} - i\vec{K} = 0$. On the other hand, the generators of L^+ are represented by Pauli matrices

$$\vec{L}^+ = \frac{1}{2}\vec{\sigma}. \tag{25}$$

Therefore we know all of them now,

$$\vec{J} = \frac{1}{2}\vec{\sigma}, \qquad \vec{K} = -i\frac{1}{2}\vec{\sigma}. \tag{26}$$

Note that the boost generators $\vec{K}$ are represented by non-Hermitian matrices, and hence the representation is non-unitary (namely $e^{i\vec{\eta}\cdot\vec{K}}$ below is not a unitary matrix).

A two-component $(\frac{1}{2}, 0)$-spinor ϕ transforms under spatial rotation by

$$\phi \to e^{i\vec{\theta}\cdot\vec{J}}\phi = e^{i\vec{\theta}\cdot\vec{\sigma}/2}\phi = \left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\frac{\vec{\theta}\cdot\vec{\sigma}}{\theta}\right]\phi. \tag{27}$$

Here, I used the notation $\theta \equiv |\vec{\theta}|$. The same spinor transforms under the Lorentz boost as

$$\phi \to e^{i\vec{\eta}\cdot\vec{K}}\phi = e^{\vec{\eta}\cdot\vec{\sigma}/2}\phi = \left[\cosh\frac{\eta}{2} + \sinh\frac{\eta}{2}\frac{\vec{\eta}\cdot\vec{\sigma}}{\eta}\right]\phi. \tag{28}$$

Here, I used the notation $\eta \equiv |\vec{\eta}|$.

4.2 $(0, \frac{1}{2})$ spinors

In this case, we set $\vec{L}^+ = \vec{J} + i\vec{K} = 0$. Again using the spin 1/2 nature, $\vec{L}^- = \frac{1}{2}\vec{\sigma}$, and hence

$$\vec{J} = \frac{1}{2}\vec{\sigma}, \qquad \vec{K} = +i\frac{1}{2}\vec{\sigma}. \tag{29}$$

The only difference from the previous case is the sign of the boost operators. Therefore we can immediately work out the transformations of $(0, \frac{1}{2})$-spinor χ

$$\chi \to e^{i\vec{\theta}\cdot\vec{J}}\chi = e^{i\vec{\theta}\cdot\vec{\sigma}/2}\chi = \left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\frac{\vec{\theta}\cdot\vec{\sigma}}{\theta}\right]\chi \tag{30}$$

under rotation (exactly the same as ϕ) while

$$\chi \to e^{i\vec{\eta}\cdot\vec{K}}\chi = e^{-\vec{\eta}\cdot\vec{\sigma}/2}\chi = \left[\cosh\frac{\eta}{2} - \sinh\frac{\eta}{2}\frac{\vec{\eta}\cdot\vec{\sigma}}{\eta}\right]\chi \tag{31}$$

under Lorentz boosts (note the minus sign in the square bracket).

One point to note is that two spin 1/2 representations are related by complex conjugation (up to unitary equivalence). Using the identity

$$(i\sigma_2)\vec{\sigma}^*(-i\sigma_2) = -\vec{\sigma}, \tag{32}$$

it is easy to see that $i\sigma_2\phi^*$ transforms the same way as χ and vice versa. Under rotations,

$$i\sigma_2\phi^* \to i\sigma_2(e^{i\vec{\theta}\cdot\vec{\sigma}/2}\phi)^* = i\sigma_2(e^{-i\vec{\theta}\cdot\vec{\sigma}^*/2}(-i\sigma_2)(i\sigma_2)\phi^* = e^{i\vec{\theta}\cdot\vec{\sigma}/2}(i\sigma_2)\phi^*, \tag{33}$$

while under boosts,

$$i\sigma_2\phi^* \to i\sigma_2(e^{\vec{\eta}\cdot\vec{\sigma}/2}\phi)^* = i\sigma_2 e^{\vec{\eta}\cdot\vec{\sigma}^*/2}(-i\sigma_2)(i\sigma_2)\phi^* = e^{-\vec{\eta}\cdot\vec{\sigma}/2}(i\sigma_2)\phi^*. \tag{34}$$

Therefore, two representations are practically complex conjugates to each other (up to the unitary transformation $i\sigma_2$).

4.3 Dirac spinors

There is a much more elegant way to obtain spin 1/2 representation using Clifford algebra,[4] as discussed in Peskin–Schroeder. This construction works in any dimensions of spactime, but it is not clear why and does not give you any other spins. For the latter purpose, you need to go back to the construction in the previous subsections.

It is a two-step process. The first step is to identify matrices that satisfy the *Clifford algebra*

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{35}$$

In four-dimensional Minkowski spacetime, Peskin–Schroeder picks

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}. \tag{36}$$

[4]For Clifford algebra and spinors in arbitrary dimensions, see my 230A lecture notes.

It is easy to verify that they satisfy the required Clifford algebra. The main point is that once γ-matrices are found, they will immediately give a representation of the Lorentz group with the generators

$$M^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{37}$$

Again, it is straightforward to verify that they satisfy the commutation relations Eq. (10).

4.4 Four-vectors

4.5 Field Strengths

5 Poincaré Group and Little Group

Lorentz transformations are not the only symmetries of the Minkowski spacetime. Translation in space and time are also symmetries. As you know already, translations are generated by energy and momentum operators $P^\mu = (E, \vec{p})$. Therefore, it is useful to include them in our discussions as well. Because all symmetries are supposed to be unitary (or anti-unitary) operators on the Hilbert space to preserve the probabilities, we now look for their unitary representations.

Obviously, translations in different directions commute:

$$[P^\mu, P^\nu] = 0. \tag{38}$$

On the other hand, the energy-momentum four-vector should transform as a four-vector, and hence

$$[M^{\mu\nu}, P^\rho] = i(g^{\nu\rho}P^\mu - g^{\mu\rho}P^\nu). \tag{39}$$

Together with the commutation relations among the Lorentz generators $M^{\mu\nu}$, they form the Pointcaré algebra.

To see how Lorentz transformation act on the Hilbert space of Quantum Field Theories, we need to represent the entire Poincaré group, not just the Lorentz group.

If you specialize them to the spatial components only in three-dimensional space, we recover familiar commutators from quantum mechanics, by identifying $J^k = \epsilon^{kl} M^{Lem}$,

$$[J^1, J^2] = [M^{23}, M^{31}] = ig^{33} M^{21} = i(-1)(-M^{12}) = iJ^3, \quad (40)$$

$$[J^1, P^2] = [M^{23}, P^2] = -ig^{22} P^3 = iP^3. \quad (41)$$

There are two Cosimo operators of the Poincareé symmetry. One is $P^2 = P^\mu P_\mu$, which obviously commutes with P^μ and is also Lorentz invariant and hence commutes with $M^{\mu\nu}$. A one-particle state is obviously its eigenstate with the eigenvalue $P^2 = m^2$. The other Cosimo operator is made of Pauli–Lubanski pseudo-vector

$$W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}. \quad (42)$$

Note that the orbital angular momentum drops out from $M_{\rho\sigma}$ because of the anti-symmetry with the momentum vector. Namely it picks up only the spin part of the angular momentum.

Commutators of W^μ with P^κ leave another P_ρ or P_σ contracted with the Levi–Civita symbol and hence vanish. It transforms as a Lorentz vector and hence has the same commutator with $M^{\mu\nu}$ as the momentum vector, but has the opposite transformation under parity. Clearly $W^2 = W^\mu W_\mu$ is Lorentz invariant and hence commutes with all generators.

For a massive particle $P^2 = m^2 > 0$, we can always go to its rest frame $P^\mu = (m, 0, 0, 0)$. We define the "little group" of the Poincaré group that preserves this four-momentum. It is nothing but the group of three-dimensional rotations $SO(3)$. We can classify states according to the $SO(3)$ eigenvalues, namely spin s and one of its component, say, s_z. Because the only non-vanishing component of the four-momentum is $P^0 = m$, only spatial components of the Pauli–Lubanski pseudo-vector survives, $\vec{W} = m\vec{s}$. Therefore, $W^2 = -\vec{W}^2 = m^2 s(s+1)$. All massive particles, therefore, can be classified by their mass and spin eigenvalues.

The situation for massless particles $P^2 = 0$ are tricky. We can always go to the frame where $P^\mu = E(1, 0, 0, 1)$. This four-momentum does not change under the rotation of the $x - y$ plane. However, note that

$$[M^{0i}, P^\mu] = -i(g^{0\mu} P^i - g^{i\mu} P^0) = ig^{i\mu} P^0, \quad (43)$$

$$[M^{3i}, P^\mu] = -i(g^{3\mu} P^i - g^{i\mu} P^1) = ig^{i\mu} P^3. \quad (44)$$

Since $P^0 = P^3 = E$, we find that $M^{0i} - M^{3i}$ remain symmetries, and hence a part of the little group. Namely, J_z, $K_x - J_y$, and $K_y + J_x$ are the generators of the little group with the commutation relations

$$[K_x - J_y, K_y + J_x] = 0, \quad [J_z, K_x - J_y] = i(K_y + J_x), \quad [J_z, K_y + J_x] = -i(K_x - J_y). \tag{45}$$

They are equivalent to the group of rotation of a plane and translations in two directions, called Euclidean motion group in two dimensions E_2.

The problem here is that E_2 is non-compact, and its unitary representations are in general infinite-dimensional. Once that we specified four-momentum of the massless particle, we don't think there should be infinite number of states left. The only way to find a finite-dimensional unitary representation is to set all non-compact generators consistently to zero. We can do so, by requiring $K_x - J_y = K_y + J_x = 0$ with no contradictions with the commutation relations. Then the only remaining generator is the rotation around the z-axis J_z, which is nothing but the *helicity* of the particle,

$$h = \frac{\vec{J} \cdot \vec{p}}{|\vec{p}|} = J_z \tag{46}$$

if $\vec{p} \parallel z$. It can have an arbitrary eigenvalue. Hence, any massless particle carries the helicity eigenvalue h in addition to its four-momentum. Then the CPT theorem says its anti-particle must carry the opposite helicity eigenvalue $-h$.

What are the allowed values of h? The discussion here does not give any additional constraints. The real constraint is that the states must appear as a result of the quantization of fields, that restricts the spin to be half-integers. Therefore, the quantum field theory only allows for half-integer helicities for massless particles.

If you want to know more about the Poincaré symmetry, its little group, and its unitary representation in arbitrary dimensions, consult my 230A lecture notes here.

A Groups

A mathematical *group* is defined by a set of rules for multiplications. For any element $g_1, g_2 \in G$, there is a product $g_1 g_2 \in G$. It is called *abelian* if the multiplication is commutative $g_1 g_2 = g_2 g_1$ for any elements, while it is *non-abelian* if some of the elements do not commute. There has to be one element called identity e which satisfies $ge = eg = g$. In addition, there must be the inverse g^{-1} for any element $gg^{-1} = g^{-1}g = e$. Then G is called a *group*.

Groups with smooth continuous parameters are *Lie groups*. Translations can be done by a little, or a lot. A succession of little translations will get to a big translation. This is an example of Lie groups.

Generators of the group correspond to infinitesimal transformations, namely group elements very close to the identity element. The multiplication rules of a group are in one-to-one correspondence to the commutation relations of the generators. The familiar examples are the momentum operator $\vec{p}$ generating the spatial translation by a finite distance $\vec{a}$ as $U(\vec{a}) = e^{i\vec{p}\cdot\vec{a}}$, or angular momentum operators $\vec{J}$ generating the spatial rotation $U(\vec{\theta}) = e^{i\vec{J}\cdot\vec{\theta}}$ around the axis $\vec{\theta}$ by the angle $|\vec{\theta}|$. The momentum operators commute among each other, and hence two translations simply add: $U(\vec{a})U(\vec{b}) = e^{i\vec{p}\cdot\vec{a}}e^{i\vec{p}\cdot\vec{b}} = e^{i\vec{p}\cdot(\vec{a}+\vec{b})} = U(\vec{a}+\vec{b})$. Namely the vanishing commutators $[p_i, p_j] = 0$ specify this property of spatial translations. On the other hand, two successive rotations are quite complicated in general. When $U(\vec{\theta}_1)U(\vec{\theta}_2) = U(\vec{\theta}_3)$, it is not easy to write $\vec{\theta}_3$ in terms of $\vec{\theta}_{1,2}$. This is a consequence of the commutation relations.

Therefore, the commutation relations among generators basically define the group. The commutation relations are called *Lie algebra*.

B Names of continuous groups

Groups with smooth continuous parameters are called Lie groups. Among them, *compact Lie groups* are the most studied. These are groups where doing a small transformation many many times will bring you back to where you started. Rotation of the Earth is a good example. On the other hand, *non-compact Lie groups* allow for a transformation to keep going indefinitely. Lorentz boost and translation in space are good examples.

There are three types of *classical groups*.

B.1 General and Special Linear Groups

The collection of all $N \times N$ real matrices with non-zero determinant (so that inverse exists) is called $GL(N, \mathbb{R})$, "general linear." If you require the matri-

$GL(N,\mathbb{C})$	$GL(N,\mathbb{R})$	$SL(N,\mathbb{C})$	$SL(N,\mathbb{R})$
$2N^2$	N^2	$2N^2-2$	N^2-1

$SO(N)$	$U(N)$	$SU(N)$	$USp(N)$	$Sp(N,\mathbb{R})$
$\frac{1}{2}N(N-1)$	N^2	N^2-1	$2N(2N+1)$	$N(2N+1)$

Table 1: The number of generators for representative classical groups.

ces to have determinant one, they form a subgroup $SL(N,\mathbb{R})$ called "special linear." The word "special" normally means determinant is one. Exactly the same works for complex numbers, defining $GL(N,\mathbb{C})$ and $SL(N,\mathbb{C})$. The names of their Lie algebras are usually written with small German letters, $\mathfrak{sl}(N,\mathbb{C})$, $\mathfrak{gl}(N,\mathbb{R})$ etc.

B.2 Orthogonal Groups

An orthogonal group $O(N)$ is a group of $N\times N$ orthogonal matrices that satisfy $OO^T=I$. The determinant of an orthogonal matrix can be either $+1$ or -1. The orthogonal matrices with determinant $+1$ form a subgroup of $O(N)$, called $SO(N)$. S stands for "special," which is supposed to mean "unit determinant." In most physics applications, orthogonal groups appear only with real coefficients, $O(N,\mathbb{R})$ or $SO(N,\mathbb{R})$, where the elements correspond to rotations of space. The only difference between the two is if you all parity-like transformations. Therefore, the Lie algebras, which by definition can refer only to the neighborhood of the origin, do not know the difference between the two. Therefore, we refer to them as $\mathfrak{so}(N)$ to make it clear that we cannot reach the parity-like transformations in $O(N)$ by exponentiating the generators.[5]

A generalization is indefinite orthogonal groups. Instead of requiring $OIO^T=I$, we introduce an indefinite metric

$$I_{p,q}=\mathrm{diag}(\underbrace{1,\cdots,1}_{p},\underbrace{-1,\cdots,-1}_{q}), \tag{47}$$

[5]Fields with half-odd spins change their signs under 2π rotations. Therefore, strictly speaking, the half-odd spins are not representations of the rotation groups, but of their *double cover*. Namely that you define an enlarged group $\mathrm{Spin}(N)$ where the 2π rotations form a $\mathbb{Z}_2$ subgroup containing only two elements $+1$ and -1. The rotation group is then a coset group $SO(N)=\mathrm{Spin}(N)/\mathbb{Z}_2$. The familiar situation is the three-dimensional rotation group $SO(3)=\mathrm{Spin}(3)/\mathbb{Z}_2$.

and require $OI_{p,q}O^T = I_{p,q}$. In other words, it is an invariance of the Minkowski-like spacetime with p space and q time directions (or vice versa). It defines again $O(p,q)$ or $SO(p,q)$ groups. The proper Lorentz group of our four-dimensional Minkowski spacetime is $SO(3,1)$, and together with parity and time-reversal, $O(3,1)$.

B.3 Unitary Groups

A unitary group $U(N)$ is a group of $N \times N$ unitary matrices that satisfy $UU^\dagger = I$. The determinant of a unitary matrix can be any phase. The unitary matrices with unit determinant $+1$ form a subgroup of $U(N)$, called $SU(N)$. In most physics applications, unitary groups appear only with complex coefficients, $U(N,\mathbb{C})$ or $SU(N,\mathbb{C})$. The difference between $U(N)$ and $SU(N)$ is that $U(N)$ contains an additional freedom of an overall phase change. Since the phases form an abelian group of "1×1 unitary matrices" $U(1)$, we see $U(N) \approx SU(N) \times U(1)$.[6]

A generalization is indefinite unitary groups. Instead of requiring $UIU^\dagger = I$, we introduce an indefinite metric

$$I_{p,q} = \mathrm{diag}(\underbrace{1,\cdots,1}_{p},\underbrace{-1,\cdots,-1}_{q}), \tag{48}$$

and require $UI_{p,q}U^\dagger = I_{p,q}$. It defines $U(p,q)$ or $SU(p,q)$ groups.

B.4 Symplectic Groups

This is a group that shows up in classical Hamiltonian mechanics when you discuss canonical transformations. We look for linear transformations among the canonical coordinates q_i and their conjugate momenta p_j while preserving Poisson brackets $\{q_i, p_j\} = \delta_{ij}$ for $i,j = 1,\cdots,N$. Namely, we define a big

[6] I used $\approx$ because this is not precise. The element $e^{2\pi i/N}$ belongs to both $SU(N)$ and $U(1)$, and generate a $\mathbb{Z}_N$ subgroup. Therefore, I need to avoid double counting and hence $U(N) = [SU(N) \times U(1)]/\mathbb{Z}_N$ to be precise.

$2N$-component column vector

$$x = \begin{pmatrix} q_1 \\ \vdots \\ q_N \\ \hline p_1 \\ \vdots \\ p_N \end{pmatrix}. \tag{49}$$

The Poisson brackets then form a big $2N \times 2N$ matrix,

$$J = (\{x_i, x_j\}) = \left(\begin{array}{c|c} 0 & I_N \\ \hline -I_N & 0 \end{array}\right). \tag{50}$$

Here, I_N is an $N \times N$ identity matrix.

Linear transformations on the vector $x \to Sx$ preserve the Poisson brackets if

$$S^T J S = J. \tag{51}$$

The matrices S form the symplectic group $Sp(2N, \mathbb{R})$.[7]

The generators appear for infinitesimal transformations $S = I + i\omega + O(\omega)^2$. Eq. (51) implies

$$\omega^T J + J\omega = 0. \tag{52}$$

Using the fact $J^T = -J$, we see $(J\omega)^T = \omega^T J^T = -\omega^T J = J\omega$. Therefore, $J\omega$ is a general $2N \times 2N$ symmetric real matrix with $N(2N+1)$ independent components. If orthogonality is further imposed, there remain N^2 generators, and the group is equivalent to $U(N)$.

If you allow for complex coefficients, $Sp(2N, \mathbb{C})$ has $2N(2N+1)$ generators. What appears often in the physics literature is $USp(2N)$, where unitarity is also required. It gives an additional requirement of hermiticity on the generators,

$$\omega^\dagger = \omega. \tag{53}$$

Then

$$J\omega^* J = J\omega^T J = J(-J\omega) = \omega. \tag{54}$$

This cuts down the number of generators to its half $N(2N+1)$.

[7] There are two camps with this notation. Since the vector space is $2N$-dimensional, some people prefer to use $2N$. On the other hand, the phase space is always even-dimensional because of (q, p) pairs and the factor of two is not a useful piece of information. Some feel N is sufficient. Just be careful which notation is being used.

B.5 Equivalence of Small Groups

Some of the classical groups are equivalent when they are small. The well-known example is between the rotation group $SO(3)$ and group of spin 1/2 $SU(2)$. To be precise, $SO(3)$ comes back to $+1$ for 2π rotation, while 2π rotation is -1 for $SU(2)$, and hence $SO(3) \cong SU(2)/\mathbb{Z}_2$. Similarly, $USp(2) = SU(2)$, $SO(4) \cong [SU(2) \times SU(2)]/\mathbb{Z}_2$, $SO(5) \cong USp(4)/\mathbb{Z}_2$, $SO(6) \cong SU(4)/\mathbb{Z}_2$.

B.6 Simple Compact Groups

Simple compact Lie algebras have been completely classified by Dynkin. The word "simple" means that the Lie algebra does not contain a generator that commutes with all the other generators, and the generators cannot be split into more than one mutually commuting sets. There are four series. A_N, B_N, C_N, D_N generates $SU(N+1)$, $SO(2N+1)$, $USp(2N)$, and $SO(2N)$, respectively. There are several simple compact Lie algebras that do not belong to any of the series and hence called "exceptional," E_6, E_7, E_8, G_2, and F_4. E_6 and E_8 appear as candidates of unified theories, G_2 in string compactifications.

References

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