AN INTRODUCTION TO QUANTUM FIELD THEORY

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Abstract

The basic concepts and ideas underlying quantum field theory are presented, together with a brief discussion of Lagrangian field theory, symmetries and gauge invariance. The use of the Feynman rules to calculate scattering amplitudes and cross sections is illustrated in the context of quantum electro dynamics.

1. INTRODUCTION

Quantum field theory provides a successful theoretical framework for describing elementary particles and their interactions. It combines the theory of special relativity and quantum mechanics and leads to a set of rules that allows one to compute physical quantities that can be compared with high energy experiments. The physical quantity of most interest in high energy experiments is the cross-section σ. It determines directly the number of events that an experimentalist will be able to see, and also is in a direct way related to the quantum mechanical probabilities that can be computed with the help of field theory.

In these notes I will present the basic concepts and ideas underlying quantum field theory. Starting from the quantum mechanics of relativistic particles I will introduce quantum field operators and explain how these are used to describe the interactions among elementary particles. These notes also contain a brief discussion of Lagrangian field theory, symmetries and gauge invariance. Finally, after presenting a heuristic derivation of the Feynman rules, I'll illustrate in the context of quantum electro dynamics how to calculate the amplitude that determines the cross section for a simple scattering process. But before we turn to the quantum mechanics, let us first discuss the meaning of the cross section in more familiar classical terms.

1.1 From Events to Cross Sections

For concreteness, let us consider an experiment with two colliding particle beams in which one beam consists of, say, electrons and the other beam of positrons. An experimentalist can now collect and count the events in which an electron and a positron collide and produce several other particles. The number of events N that occur in a certain amount

of time depends on various experimental details, in particular the densities ρ_1 and ρ_2 of the particles inside the beams and of course the particle velocities $\vec{v}_1$ and $\vec{v}_2$.

To find out how these parameters affect the number of events, let us first consider the probability that an event occurs inside a a small volume ΔV and within a time interval Δt. In this case we can assume that the densities and velocities are constant. Now imagine that one of the particles, say the electron, has a finite cross-section σ as seen from the perspective of the other particle, the positron. After a time Δt each electron has moved with respect to the positrons over a distance $|\vec{v}_1 - \vec{v}_2|\Delta t$, where $\vec{v}_1 - \vec{v}_2$ is the relative velocity, and in this way it has swept out a part of space with a volume equal to $\sigma|\vec{v}_1 - \vec{v}_2|\Delta t$. The probability that in this time the electron has collided with a positron is equal to the number of positrons inside this volume, which is $\rho_2\sigma|\vec{v}_1 - \vec{v}_2|\Delta t$. Finally, to obtain the total number ΔN of collisions we have to multiply this with the number of electrons inside the volume ΔV; this number is $\rho_1\Delta V$. In this way we find that the number of events that occur in ΔV and Δt is

$$\Delta N = \sigma\rho_1\rho_2|\vec{v}_1 - \vec{v}_2|\Delta V\Delta t \tag{1}$$

In a realistic situation the densities $\rho_i(\vec{x}, t)$ of the particles and their velocities are not constant in space and time. Instead of the velocity $\vec{v}$ it is also more appropriate to use the current density $\vec{j}$ that describes the flux of the particles per unit of area. It is defined by $\vec{j} = \rho\vec{v}$ where $\vec{v}$ is the local velocity at a certain point in space and time. By repeating the above argumentation we can express the number of events in an infinitesimal volume dV and time interval dt as $dN = \sigma|\rho_2\vec{j}_1 - \rho_1\vec{j}_2|d^3\vec{x}dt$. Then by integrating of space and time we obtain the following expression for the total number of events N

$$N = \sigma \int dt\, L(t), \tag{2}$$

where[1)]

$$L(t) = \int d^3x|\rho_1(\vec{x}, t)\vec{j}_2(\vec{x}, t) - \rho_2(\vec{x}, t)\vec{j}_1(\vec{x}, t)|. \tag{3}$$

All experimental details that influence the number of events are combined in the time integral of $L(t)$, which is called the integrated luminosity. The integrated luminosity does not depend on the kind of process that one is interested in, and so it can be measured by looking at a reference process for which the cross section σ is accurately known. With this knowledge one can then determine the cross sections for all other processes. These cross-sections σ are independent of the experimental details, and thus can be used to compare the data that come from different experiments. They are also the relevant quantities that one wishes to compute using quantum field theory.

1.2 Quantum Mechanical Probabilities

Just to give an idea of how the cross section σ is related to a quantum mechanical amplitude, let us assume that before the collision the densities and velocities of the particles inside the beams are accurately known. In terms of quantum mechanics this means

1) A more careful analysis shows that the integrand has an additional relativistic contribution $\frac{1}{c}|\vec{j}_1 \times \vec{j}_2|$. This term can be dropped in a situation where $\vec{j}_1$ and $\vec{j}_2$ are (approximately) parallel.

that one has specified the initial quantum state $|\Psi_i\rangle$ of the colliding particles. Similarly, by measuring the energies and directions of the particles that are being produced by the collisions one selects a certain final wave-function $|\Psi_f\rangle$ that describes the state of the out-going particles. According to the rules of quantum mechanics the probablity W for this event to happen is given by the absolute valued squared of the overlap between the final state and the initial state

$$W_{i\to f} = |\langle\Psi_i|\Psi_f\rangle|^2$$

Quantum field theory provides the tools to compute the amplitudes $\langle\Psi_i|\Psi_f\rangle$, and hence determine the quantum mechanical probabilities W. In fact, the quantity that is most directly computed is the probability W that a collision takes place per unit of volume and time for a situation with constant particle densities ρ_1 and ρ_2. For this situation W may be identified with the ratio $\Delta N/\Delta V\Delta t$, which according to equation (1) gives the cross section σ times the product of the densities and relative velocity. All this will be explained in more detail in the following sections.

2. RELATIVITY AND QUANTUM MECHANICS

The elementary particles that are used in high energy experiments have velocities close to the speed of light, and should therefore be described by the theory of special relativity. For relativistic particles it is convenient to specify their motion, instead by their velocity $\vec{v}$, in terms of their momentum

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-(\frac{\vec{v}}{c})^2}} \tag{4}$$

An important reason for this is that during collision the total momentum of the particles will be conserved. The total energy of the particles will also be conserved. In special relativity the energy of a particle with momentum $\vec{p}$ is given by

$$E = \sqrt{\vec{p}^2c^2 + m^2c^4} \tag{5}$$

For a particle at rest this becomes Einstein's famous relation $E = mc^2$.

2.1 Relativistic Wave Functions

In quantum mechanics the energy E becomes identified with the Hamiltonian operator H. The time evolution of the wave function for a single particle is, as usual, governed by the Schrödinger equation in which the Hamiltonian H acts as a differential operator. The form of this differential operator can be read of from (5) by replacing the momentum $\vec{p}$ by the operator $\vec{\mathbf{p}} = -i\hbar\vec{\nabla}$. This gives

$$i\hbar\partial_t\psi(\vec{x},t) = H\psi(\vec{x},t) \tag{6}$$

with

$$H = \sqrt{-\hbar^2\vec{\nabla}^2 + m^2c^4}. \tag{7}$$

An important difference with the non-relativistic case is, however, that the Hamiltonian involves a square root. Still the Schrödinger equation can easily be solved by expanding the wave-function in terms of plane waves

$$\psi_{\vec{k}}(\vec{x},t) = e^{i\vec{k}\cdot\vec{x}-i\omega(\vec{k})t}.$$

These plane waves are the eigenfunctions of the momentum operators $\vec{\mathbf{p}}$ with eigenvalue $\vec{p} = \hbar\vec{k}$ and are also energy eigenstates with eigenvalue $E = \hbar\omega(\vec{k})$. The most general solution to the Schrödinger equation can now be written as[2)]

$$\psi(\vec{x},t) = \frac{1}{(2\pi)^3}\int \frac{d^3\vec{k}}{2\omega(\vec{k})}\, e^{i\vec{k}\cdot\vec{x}-i\omega(\vec{k})t}\,\psi(\vec{k}), \tag{8}$$

where

$$\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}. \tag{9}$$

To simplify the formulas we put here and in the following $\hbar = c = 1$. When the momentum $\vec{p}$ is well specified the wave-function $\psi(\vec{k})$ will be strongly peaked near $\vec{k} = \vec{p}$. The position of the particle will in this case not be very precisely determined due to the uncertainty principle. So the wave function gives us only a probability $\rho(\vec{x},t)$ for finding a particle at a position $\vec{x}$ at a certain time t. Because probabilities have to be conserved, there must also exist a probability current density $\vec{j}(\vec{x},t)$ that obeys the conservation law

$$\partial_t\rho(\vec{x},t) = \vec{\nabla}\cdot\vec{j}(\vec{x},t) \tag{10}$$

For the relativistic wave function the probability density and current density are given by the expressions

$$\begin{aligned}\rho &= \frac{i}{2m}(\psi^*\partial_t\psi - \psi\partial_t\psi^*)\\ \vec{j} &= -\frac{i}{2m}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)\end{aligned} \tag{11}$$

These densities can in fact be identified with the particle densities ρ and $\vec{j}$ inside the beam. Notice that for a plane wave $\psi_{\vec{k}}$ these densities are given by $\rho = \frac{\omega}{m}$ and $\vec{j} = \frac{\vec{k}}{m}$. To verify that these expressions indeed satisfy the conservation law (10) one has to use the 'square' of the Schrödinger equation (6)

$$(\partial_t^2 - \vec{\nabla}^2 + m^2)\psi(\vec{x},t) = 0 \tag{12}$$

This equation is called the Klein-Gordon equation. We note, however, that the Klein-Gordon equation allows more general solutions than the original Schrödinger equation. Namely, if we change the sign of the time frequency $\omega(\vec{k}) \to -\omega(\vec{k})$ in the expression (8) it would still be a solution of (12). These negative frequency solutions are not physically acceptable as wave functions because they would correspond to particles with negative energy.

2.2 The Dirac Equation

For particles with spin the wave function not only carries information about the momentum and energy but also specify the polarization of the particle. An electron has spin 1/2, and so its wave function has two independent components $\chi_\uparrow(\vec{x},t)$, for spin up,

2) Here the normalization factor $2\omega(\vec{k})$ is chosen so that the function $\psi(\vec{k})$ has a relativistic invariant meaning.

and and $\chi_\downarrow(\vec{x},t)$, for spin down. Both components satisfy the Schrödinger equation (6) and through a similar expansion as (8) lead to two independent functions $\chi_\uparrow(\vec{k})$ and $\chi_\downarrow(\vec{k})$ that can be combined in a two component "spinor"

$$\chi(\vec{k}) = \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix}(\vec{k}) \tag{13}$$

In the rest frame of the electron the direction of the spin is measured by the Pauli matrices σ_i. These are 2×2 matrices satisfying $\sigma_i^2 = \mathbf{1}$ and $\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3$, etc.

Dirac discovered a relativistic wave equation for a spin $\frac{1}{2}$ particle in terms of a 4-component wave-function ψ_a. It is convenient to first decompose this four component spinor in two two-component spinors ψ_+ and ψ_-.

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

and introduce the matrices

$$\vec{\alpha} = \begin{pmatrix} \mathbf{0} & \vec{\sigma} \\ \vec{\sigma} & \mathbf{0} \end{pmatrix} \qquad \text{and} \quad \beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$$

The Dirac equation can then be given in the form of a Schrödinger equation

$$i\partial_t\psi = (-i\vec{\alpha}\cdot\nabla + m\beta)\psi \tag{14}$$

One of the motivations of Dirac to write down this wave equation is that the associated probability density ρ and the probability current $\vec{j}$ do not involve time or spatial derivatives, but take the simple form

$$\rho = \psi^\dagger\psi, \qquad \vec{j} = \psi^\dagger\vec{\alpha}\psi. \tag{15}$$

Using the Dirac equation one easily verifies that these current densities satisfy the conservation law (10).

It follows from (14) that the separate components of ψ_a satisfy the Klein-Gordon equation. The wave functions that describe physical particles must again have positive energies $\omega(\vec{k}) > 0$. Hence, the physical solutions to the Dirac equation can be expanded in plane waves as

$$\psi(\vec{x},t) = \frac{1}{(2\pi)^3}\int \frac{d^3\vec{k}}{2\omega(\vec{k})}\, e^{i\vec{k}\cdot\vec{x} - i\omega(k)t}\, u(\vec{k}) \tag{16}$$

where

$$u(\vec{k}) = \sqrt{\tfrac{\omega+m}{2m}}\begin{pmatrix} \chi \\ \frac{\vec{k}\cdot\vec{\sigma}}{\omega+m}\chi \end{pmatrix} \tag{17}$$

and $\chi(\vec{k})$ is the two component spinor introduced above. Notice that for an electron at rest the momentum $\vec{k} = 0$ and energy $\omega(\vec{k}) = m$. As a consequence, the lower two components of u_a vanish while the upper two components are identified with the spinor χ. This shows that the Dirac equation indeed describes spin $\frac{1}{2}$ particles with two polarizations.

2.3 Lorentz Covariant Notation

The invariance under Lorentz transformations of the Klein-Gordon and Dirac equations can be made more manifest by introducing a covariant notation for the coordinates, $x^\mu = (\vec{x}, t)$, where $\mu = 1, 2, 3, 0$ and the derivatives $\partial_\mu = (\vec{\nabla}, \partial_t)$. We also introduce a metric $\eta_{\mu\nu}$ with signature $\mathrm{diag}(-1, -1, -1, +1)$ that is used to raise, lower, and contract indices. The Klein-Gordon equation can in this notation be written as

$$(\partial_\mu \partial^\mu + m^2)\varphi(x) = 0 \tag{18}$$

Here we changed the notation for the wave function to φ to distinguish it from the Dirac wave-function. The conserved probability current $j_\mu = (\vec{j}, \rho)$ becomes in covariant notation $j_\mu = \frac{i}{2m}(\varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^*)$.

To write the Dirac equation in covariant form we define the Dirac gamma-matrices $\gamma_\mu = (\vec{\gamma}, \gamma_0)$ by $\vec{\gamma} = \beta\vec{\alpha}$ and $\gamma_0 = \beta$. These gamma matrices satisfy the anti-commutation relations $\{\gamma_\mu, \gamma_\nu\} \equiv \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}$. With these definitions one easily verifies that the Dirac equation (14) coincides with

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \tag{19}$$

The conserved probability current (15) is in covariant notation $j_\mu = \overline{\psi}\gamma_\mu\psi$, where $\overline{\psi} = \psi^\dagger\gamma_0$.

3. QUANTUM FIELDS AND THE FOCK SPACE

To describe the processes that take place in high energy collisions it is not sufficient to restrict ones attention to wave functions of single particles. These processes involve several particles which at early times before the collision and also after the collisions are at different positions $\vec{x}_i$ and may carry different momenta $\vec{k}_i$. As long as the particles are well separated in space we can represent the n-particle wave function by a product of the single particle wave-functions

$$\Psi(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n, t) = \psi(\vec{x}_1, t)\psi(\vec{x}_2, t)\ldots\psi(\vec{x}_n, t)$$

In this regime the total energy is approximately equal to the sum of the energies of the individual particles. The time evolution outside the interaction region may therefore be described by the free Hamiltonian

$$H_0 = \sum_{i=1}^{n} \sqrt{-\vec{\nabla}_i^2 + m_i^2}$$

In non-relativistic quantum mechanics we are used to describe interactions among particles by introducing a potential $V(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n)$ in the Hamiltonian. However, this prescription can not be valid in a relativistic theory because it would describe an instantaneous force between particles that are outside each others light-cone. As we will see, the correct way to describe interactions is to allow the particles to emit and absorb other particles. Through these interactions the number of particles may change in time. Indeed, we know that a collision of a electron and positron may result in the production of many other particles. This is what makes quantum field theory different from usual non-relativistic quantum mechanics in which the number of particles is always fixed. The physical explanation of

this fact is that the relation $E = mc^2$ allows the center of mass energy of the colliding particles to be used to create new particles. We may conclude from this that quantum field theory needs to be formulated in a Hilbert space that contains quantum states with arbitrary number of particles.

3.1 The Fock Space

The Hilbert space that consist of all multi-particle states is called a Fock space. We will now explain how to construct this Fock space. First, there exists one unique state without particles which is called the vacuum state. It is denoted by $|vac\rangle$ and is normalized such that $\langle vac|vac\rangle = 1$. Then there are states $|\vec{k}\rangle$ that describe one particle with momentum $\vec{k}$. We normalize these one-particle states as

$$\langle \vec{k}'|\vec{k}\rangle = 2\omega(\vec{k})(2\pi)^3\,\delta^{(3)}(\vec{k}-\vec{k}'). \tag{20}$$

The quantum state for n particles with momenta $\vec{k}_1$, $\vec{k}_2$,...,$\vec{k}_n$ is denoted by

$$|\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n\rangle$$

These states have a normalization similar to (20) including factors of $2\omega(\vec{k}_i)(2\pi)^3$.

In quantum field theory it is possible to have transitions between states with different number of particles. These transitions are described with the help *creation* operators $a_k^\dagger$. When the creation operator $a_k^\dagger$ acts on the vacuum state $|vac\rangle$ it gives the one-particle state $|\vec{k}\rangle$. More generally, by acting with $a_k^\dagger$ on a state with n particles one adds another particle with momentum $\vec{k}$, and hence one obtains a state with $n+1$ particles. In this way one concludes that any n-particle state is obtained from the empty vacuum by acting successively with n creation operators

$$|\vec{k}_1, \ldots, \vec{k}_n\rangle = a^\dagger_{\vec{k}_1} \ldots a^\dagger_{\vec{k}_n}|vac\rangle. \tag{21}$$

Here we assumed for simplicity that all momenta $\vec{k}_i$ are different. Similarly we can define so-called *annihilation* operators $a_{\vec{k}}$ that reduce the number of particles with momentum $\vec{k}$ by one. When these annihilation operators act on a state that contains no particle with momentum $\vec{k}$ one gets zero. In particular, the vacuum state $|vac\rangle$ satisfies $a_{\vec{k}}|vac\rangle = 0$. The creation and annihilation operators obey the commutation relations

$$[a^\dagger_{\vec{k}}, a_{\vec{k}'}] = 2\omega(\vec{k})(2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}'), \tag{22}$$

which is similar to the algebra of the creation and annihilation operators for the harmonic oscillator.

3.2 Quantum Fields

As its name suggests, the central objects in quantum field theory are the *quantum fields*. A quantum field is an (hermitean) operator $\phi(\vec{x},t)$ that at time t destroys or creates a particle at the position $\vec{x}$. More precisely, by acting with the quantum field $\phi(\vec{x},t)$ on the empty vacuum we obtain a state with one particle

$$\phi(\vec{x},t)|vac\rangle = \frac{1}{(2\pi)^3}\int \frac{d^3\vec{k}}{2\omega(\vec{k})}\, e^{i\vec{k}\cdot\vec{x}-i\omega(\vec{k})t}|\vec{k}\rangle \tag{23}$$

Here we notice the similarity with the expression (8) for the single particle wave function. But it should be clear that quantum field $\phi(\vec{x},t)$ has quite a different interpretation than the wave function $\psi(\vec{x},t)$. By expanding the field $\phi(x,t)$ in terms of plane waves as

$$\phi(\vec{x},t) = \frac{1}{(2\pi)^3}\int \frac{d^3\vec{k}}{2\omega(\vec{k})}\left(a^\dagger_{\vec{k}} e^{i\vec{k}\cdot\vec{x}-i\omega(k)t} + a_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}+i\omega(k)t}\right) \tag{24}$$

one obtains the creation and annihilation operators $a^\dagger_{\vec{k}}$ and $a_{\vec{k}}$ that we introduced before. Notice that the quantum field $\phi(x,t)$ satisfies the Klein-Gordon equation (12), but that unlike the wave-function $\psi(\vec{x},t)$ it contains in its expansion plane waves with negative time frequency $-\omega(\vec{k})$.

The canonical commutation relations between the creation and annihilation operators imply that the quantum field $\phi(\vec{x},t)$ does not necessarily commute with the field $\phi(\vec{y},t')$ at some other point in space and time. However, two operators that act at the same time $t'=t$ but at different points in space should commute. Otherwise there would be a contradiction with one of the postulates of relativity that no signal can travel faster than light. Indeed one can show that

$$[\pi(\vec{y},t),\phi(\vec{x},t)] = -i\delta^{(3)}(\vec{x}-\vec{y}) \tag{25}$$

where $\pi(\vec{y},t) = \partial_t\phi(\vec{y},t)$ is called the canonically conjugate field of $\phi(\vec{x},t)$ because their relation is similar to that of the momentum $\vec{p}$ and the coordinate $\vec{x}$. We remark that by expanding the field ϕ and π in plane waves one recovers from (25) all the commutation relations (22) of the creation and annihilation operators.

3.3 Quantum Fields for Particles with Charge and Spin

The field that we just described is the simplest example of a relativistic quantum field. It describes an uncharged particle with zero spin such as the pion π^0. We will now briefly describe the quantum fields for particles with charge and spin. An important fact about Nature is that all charged particle have an associated anti-particle with an identical mass and spin but with opposite charge. For example, the anti-particle for the π^+ pion is the negatively charged pion π^-, and the positron is the anti-particle associated with the electron. The quantum field for the charged pions is a complex scalar with expansion

$$\varphi(\vec{x},t) = \frac{1}{(2\pi)^3}\int \frac{d^3\vec{k}}{2\omega(\vec{k})}\left(a^\dagger_{\vec{k}} e^{i\vec{k}\cdot\vec{x}-i\omega(k)t} + b_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}+i\omega(k)t}\right) \tag{26}$$

where $a^\dagger_{\vec{k}}$ creates a π^+ while $b_{\vec{k}}$ annihilates a π^-. The other operators $b^\dagger_{\vec{k}}$ and $a_{\vec{k}}$ are contained in the hermitean conjugate field $\varphi^\dagger$. The electron and positron are described by a complex Dirac spinor which has four components satisfying the Dirac equation. The creation and annihilation operators are again found by expanding the Dirac field in Fourier modes

$$\psi(\vec{x},t) = \frac{1}{(2\pi)^3}\int \frac{d^3\vec{k}}{2\omega(\vec{k})}\left(c^\dagger_{\vec{k},a} u^a(\vec{k}) e^{i\vec{k}\cdot\vec{x}-i\omega(k)t} + d_{\vec{k},b} v^b(\vec{k}) e^{-i\vec{k}\cdot\vec{x}+i\omega(k)t}\right) \tag{27}$$

where $u^a(\vec{k})$ is defined in (17), with $a=\uparrow$ or $\downarrow$, and $v^b(\vec{k})$ is given by a similar expression with the upper and lower components interchanged. The first term gives the creation

operators $c^\dagger_{\vec{k},a}$ for the electron and the last term describes the annihilation operators $d_{\vec{k},b}$ for the positron. Again the hermitean conjugate field gives the other operators. Because the electron and positron are fermions the creation and annihilation operators satisfy canonical *anti-commutation* rules

$$\left\{c^\dagger_{\vec{k},a}, c_{\vec{k}',a'}\right\} = \delta_{aa'} 2\omega(k)(2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}') \tag{28}$$

This will incorporate the Pauli exclusion principle, and leads, as we will see, to some slight differences in the Feynman rules for fermions compared to bosons.

3.4 The Feynman Propagator

The forces between elementary particles are often described by the mediation of another particle which is created at one point $(\vec{x},t)$ in space and time and then annihilated at another point $(\vec{x}',t')$ with $t'>t$. The quantum mechanical amplitude for such a successive creation and annihilation is given by the *propagator*

$$\Delta(\vec{x}-\vec{x}',t-t') = \langle vac|\phi(\vec{x}',t')\phi(\vec{x},t)|vac\rangle \qquad \text{for} \quad t'>t \tag{29}$$

For $t>t'$ one has to change the order of the two fields, because the operator that corresponds to the earliest point in time must also acts first on the vacuum. So in this case the particle is first created at the position $(\vec{x}',t')$ and than annihilated at $(\vec{x},t)$.

The propagator will play an important role in the Feynman rules. But just as a preparation, we will describe here how to evaluate it. First we insert the equation (23) and then use the normalization (20). In this way we obtain

$$\Delta(\vec{x}-\vec{x}',t-t') = \frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega(\vec{k})}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')-i\omega(\vec{k})|t-t'|} \tag{30}$$

Notice that the propagators only depends on the differences in the coordinates. We can use this to put $\vec{x}'$ and t' equal to zero. The above expression is a direct generalization of the familiar Yukawa potential $e^{-|m||\vec{x}|}/|\vec{x}|$ to which it would reduce in the static limit. We can rewrite the propagator in a manifestly Lorentz invariant form by using the identity

$$\frac{i}{2\pi}\int_{-\infty}^{+\infty} dk_0\, \frac{e^{-ik_0 t}}{k_0^2-\omega^2+i\epsilon} = \frac{e^{-i\omega|t|}}{2\omega}$$

which is proven by a complex contour deformation. The Feynman propagator can thus be written in covariant notation as

$$\Delta(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x}\Delta(k) \tag{31}$$

with

$$\Delta(k) = \frac{i}{k^2-m^2+i\epsilon} \tag{32}$$

and $k_\mu = (\vec{k},k_0)$ and $x^\mu = (\vec{x},t)$. In a similar way one can derive the Feynman propagator for the Dirac field. The only difference is that we have a sum over polarization states of the electron. From the normalization of the spinor $u^a(\vec{k})$ given in section 2.2 one finds that as a result the integrand contains an additional factor

$$\sum_a u_a(\vec{k})\overline{u}_a(\vec{k}) = (k_\mu\gamma^\mu + m) \tag{33}$$

where γ^μ are the gamma matrices introduced in section 2.3.

4. AMPLITUDES AND CROSS SECTIONS

In this section we will explain how the cross section is related to quantum mechanical amplitude, and we will describe how these amplitudes are represented in quantum field theory

4.1 From Amplitudes to Cross Sections

Consider an event where two particles with momenta $\vec{p}_1$ and $\vec{p}_2$ collide and produce n particles with momenta $\vec{k}_1$, ..., $\vec{k}_n$. Before the collision we can represent the wave functions of both particles by planes waves $\psi_{\vec{p}_1}$, and $\psi_{\vec{p}_2}$. We will denote this two-particle state by $|\vec{p}_1, \vec{p}_2\rangle_i$. Similarly, after the collision we have n particles with wave functions given by plane waves. The corresponding n-particle state is $|\vec{k}_1, \ldots, \vec{k}_n\rangle_f$. The quantum mechanical probability for this process is determined by the transition amplitude

$$\mathcal{A}_{i\to f}(p_1, p_2; k_i) = {}_i\langle \vec{p}_1, \vec{p}_2 | \vec{k}_1, \ldots, \vec{k}_n \rangle_f \tag{34}$$

These transition amplitudes are the objects that one would like to compute with quantum field theory. But before we discuss that in some detail, let us first explain in what way these amplitudes are related to the cross section, which after all is the observable that can be measured.

During a collision there are various physical quantities that are conserved. In particular, the total energy and momentum before and after the collision must be the same. This tells us that the transition amplitude $\mathcal{A}$ will vanish unless

$$\sum_{i=1}^{n} \omega_i(k_i) = \omega_1(p_1) + \omega_1(p_2)$$

$$\sum_{i=1}^{n} \vec{k}_i = \vec{p}_1 + \vec{p}_2$$

Examples of other conserved quantities are charge, lepton number etc. Now according to the standard rules, we can now calculate the probability by taking the square of the amplitude. Notice, however, since we have fixed the momentum of the produced particles, we are only calculation a partial probability for producing particles with momenta between $\vec{k}_i$ and $\vec{k}_i + d\vec{k}_i$. This transition probability is

$$dW(\vec{p}_1, \vec{p}_2; \vec{k}_i) = (2\pi)^4 |\mathcal{A}(\vec{p}_1, \vec{p}_2; \vec{k}_i)|^2 \prod_i \frac{d^3 k_i}{(2\pi)^3 2\omega(\vec{k}_i)} \tag{35}$$

To obtain the full probability one has to integrate of the final momenta $\vec{k}_i$. Implicit in the expression (35) are the delta-functions for momentum and energy conservation. To make them explicit one should multiply the right hand side by

$$(2\pi)^4\, \delta^{(3)}\Big(\textstyle\sum_i \vec{k}_i - \vec{p}_1 - \vec{p}_2\Big)\, \delta\Big(\textstyle\sum_i \omega(\vec{k}_i) - \omega_1 - \omega_2\Big)$$

This transition rate dW depends on the normalization of the wave functions of the two incoming particles. Because these wave-functions are represented by planes waves the associated particle densities are $\rho_i = \frac{\omega_i(\vec{p}_i)}{m_i}$ and $\vec{j}_i = \frac{\vec{p}_i}{m_i}$. Now, to go from the transition probability to the differential cross-section $d\sigma$ we can follow the same steps of section 1.1.

The luminosity is given by the integral of $|\vec{j}_1\rho_2 - \vec{j}_2\rho_1| = |\vec{p}_1\omega_2 - \vec{p}_2\omega_1|/m_1m_2$. If we divide the transition rate dW by this factor we obtain the cross section. Thus we obtain

$$d\sigma = \frac{m_1m_2}{|\vec{p}_1\omega_2 - \vec{p}_2\omega_1|} dW(p_1, p_2; k_i) \tag{36}$$

This expression is valid in the center of mass frame. To extend it to general Lorentz frames we simply note that numerator can be written in covariant form as $\sqrt{(p_1 \cdot p_2)^2 - m_1^2m_2^2}$. In the following we will not really be concerned with calculating explicitly these cross sections. Instead we will focus on the quantum mechanical amplitudes $\mathcal{A}$ and the probability rate dW.

4.2 The Interaction Hamiltonian

In quantum mechanics the time evolution of a quantum state is described by a Schrödinger equation. For the situation of our interest the Hamiltonian H is an operator that acts in the Fock space of many particles, and can be written a sum of the free Hamiltonian H_0 and an additional term H_{int} that represent the interactions among the particles. What can we say about H_{int}? Because the interactions may change the number of particles the Hamiltonian will contain operators that can create or destruct particles. Furthermore, the interactions should be consistent with causality, which means that the presence of a particle can be felt by another particle only when they have been in causal contact. This is guaranteed when the interactions that are contained in the Hamiltonian only occur between particles that are at the same point in space and time.

To make a long story short, the appropriate way to construct a Hamiltonian that satisfies all the required properties is to express the interaction Hamiltonian as an integral over space of a density $\mathcal{H}(\vec{x}, t)$. This Hamiltonian density $\mathcal{H}$ must be given in terms of a product of the quantum fields ϕ and π and, possibly, the first order spatial derivatives $\vec{\nabla}\phi$.

$$H_I(t) = \int d^3x\, \mathcal{H}_{int}\left(\pi, \vec{\nabla}\phi, \phi\right) \tag{37}$$

In fact, also the free Hamiltonian is of this form with an integrand that is quadratic in the quantum fields

$$H_0 = \int d^3x\, \left(\pi^2 + (\nabla\phi)^2\right) \tag{38}$$

This is what distinguishes the free Hamiltonian from the interaction Hamiltonian, which is an higher order operator. A typical example of an interaction term is $\mathcal{H}_{int}(\vec{x}, t) = \lambda\phi^4(\vec{x}, t)$.

In modern approaches to quantum field theory one does not often use the Hamiltonian density $\mathcal{H}$, but instead one formulates the theory in terms of a Lagrangian density $\mathcal{L}(\vec{x}, t)$. It is related to the Hamiltonian density by

$$\begin{aligned} \mathcal{H}(\pi, \phi) &= \pi\dot{\phi} - \mathcal{L}(\phi, \dot{\phi}) \\ \pi &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} \end{aligned} \tag{39}$$

The advantage of using the Lagrangian instead of the Hamiltonian formalism is that all symmetries, like Lorentz invariance, are manifest. This is because, instead of the Hamiltonian H which is an integral over space, one works with an action S that is written as an integral over space and time of the Lagrangian density $\mathcal{L}$. In the next section we will discuss the Lagrangian formulation in more detail.

4.3 The S-Matrix

We will now describe how the quantum mechanical probability amplitudes

$$\mathcal{A}(\vec{p}_i, \vec{p}_2; \vec{k}_j) = {}_i\langle \vec{p}_1, \ldots, \vec{p}_m | \vec{k}_1, \ldots, \vec{k}_n \rangle_f$$

are represented in terms of quantum field theory. The idea is as follows. We consider a quantum state $|\Psi(t)\rangle$ which at the time $t = -\infty$ coincides with the initial state. Then by studying its time evolution we can in principle find out what this state looks like at $t = +\infty$. The probability for finding the n particles in the final state $|\Psi(\infty)\rangle$ is then determined by the overlap of the n-particle state with this final state.

So, once we know the Hamiltonian density $\mathcal{H}_{int}$ in principle all we have to do is to solve the Schrödinger equation

$$i\partial_t |\Psi(t)\rangle_S = (H_0 + H_{int}) \, |\Psi(t)\rangle_S \tag{40}$$

Without interaction this would have been easy, because the free Hamiltonian just evolves each of the particles according to the usual one particle Schrödinger equation. In particular, it doesn't change the number of particles.

The non-trivial part of the time evolution is contained in the interaction Hamiltonian. One can describe the quantum states in a different representation which only sees the interactions by undoing the time evolution that is generated by the free Hamiltonian. So we introduce a new quantum state by

$$|\Psi(t)\rangle_I = e^{itH_0} |\Psi(t)\rangle_S. \tag{41}$$

The time evolution of this state is generated only by the interaction Hamiltonian

$$i\partial_t |\Psi\rangle_I = H_{int}(t) |\Psi\rangle_I \tag{42}$$

The interaction Hamiltonian $H_{int}(t)$ is expressed as in (37). Note, however, that in this picture the fields $\phi(\vec{x}, t)$ and hence $H_{int}(t)$ are time dependent. This time dependence simply means that the fields $\phi(\vec{x}, t)$ satisfy the Klein-Gordon, or Dirac, or some other linear wave equation. Formally we can write down the solution to the Schrödinger equation (40) as

$$|\Psi(t_f)\rangle_I = \mathrm{T} \exp\left[-i \int_{t_i}^{t_f} dt \, H_I(t)\right] \, |\Psi(t_i)\rangle_I \tag{43}$$

where the time ordening symbol T implies that the operators $H_I(t)$ must act successively on the state $|\Psi(t_i)\rangle$ in the order of the time t in their argument. More explicitly,

$$\mathrm{T} \exp\left[-i \int_{t_i}^{t_f} dt \, H_I(t)\right] \equiv \sum_{n=0}^{\infty} (-i)^n \int_{t_i < t_1 < t_2 < \ldots < t_n < t_f} dt_1 dt_2 \ldots dt_n \; H_I(t_n) \ldots H_I(t_2) H_I(t_1) \tag{44}$$

Next we insert the expression (37) for the interaction Hamiltonian and send $t_i \to -\infty$ and $t_f \to \infty$ to get

$$|\Psi(\infty)\rangle_I = \mathrm{T} \exp\left[-i \int d^4x \, \mathcal{H}_{int}(x)\right] \, |\Psi(-\infty)\rangle_I \tag{45}$$

where $d^4x = dt d^3x$ and $x = (\vec{x}, t)$. This exponential operator on the right-hand-side is called the S-matrix. Its matrix elements give the probability amplitude that for an initial

n-particle state $|p_1, \ldots, p_n\rangle$ the outgoing state is given by the m-particle state $|k_1, \ldots, k_m\rangle$. This amplitude is

$$\mathcal{A}(p_i; k_j) = \langle k_1, \ldots, k_m | \mathrm{T} \exp\left[-i \int d^4x \, \mathcal{H}_{int}(x)\right] |p_1, \ldots, p_n\rangle \tag{46}$$

From this expression it is straightforward to derive the Feynman rules for a given Hamiltonian densite $\mathcal{H}_{int}(x)$.

4.4 Feynman Rules

To evaluate these amplitudes one has to use perturbation theory. One expands the exponent, taking in to account the time ordening, and than one inserts the explicit expression of the interaction Hamiltonian in terms of the fields. As mentioned, the Hamiltonian density is some polynomial in the fields ϕ and its derivatives. These fields create or destroy particles at a certain position $(\vec{x}, t)$ in space and time. The power of the fields tells us how many particles are destroyed or annihilated at one point.

The various contributions to the scattering amplitude can be represented graphically in terms of Feynman diagrams. Given a diagram there is a very precise prescription to write down the expression for the contribution in terms of the propagators and vertices of a given quantum field theory. Here we will here just sketch the basic idea. Each Feynman diagram has a number of external lines that represent the incoming and outgoing particles. For the above amplitude the number of lines would be m incoming and n outgoing. We label these lines by the associated momenta k_i and p_i. According to the Feynman rules one has to write down for each of the lines the appropriate single particle wave functions describing a particle with external momentum $\vec{k}_i$ or $\vec{p}_i$. Next, the external lines come together at *vertices*. These vertices are in one to one correspondence with the various terms in the interaction Hamiltonian. For a simple situation in which $\mathcal{H}_{int}$ has just one term $\lambda\phi^4$ there is just one type of vertex where 4 lines come together. For each vertex one has to write down one power of the coupling constant λ times an i that comes from the i in front of the Hamiltonian.

Further, the sum of all momenta that flow towards a vertex must vanish. It is not necessary that all lines that come together in one vertex correspond to external lines. A Feynman diagram has also internal lines that correspond to the *propagators*. Just like the external lines, the internal lines also carry momentum. For each propagator one has to write down a factor that depends on its momentum. The precise form of this propagator depends on the type of field. For the Klein Gordon scalar field that we considered so far the propagator is

$$\Delta(k) = \frac{i}{k^2 - m^2 + i\epsilon} \tag{47}$$

where $k = (\vec{k}, k_0)$ and $k^2 = k_0^2 - \vec{k}^2$. We have already explained in section 3.4 how this expression is derived. For diagrams without closed loops the condition that momentum is conserved at each vertex determines the momenta for all the internal lines in terms of the external momenta k_i. But for diagrams with ℓ closed loops there will be ℓ undetermined momenta that have to be integrated over. Fermion loops have an additional factor of (-1) because of fermi statistics. Finally, one has to sum over all inequivalent ways that the external lines can be connected through propagators and vertices. What I have just given is only a rough sketch of the Feynman rules. The precise rules depend of course on the quantum field theory that one considers and, unfortunately, also on conventions.

5. LAGRANGIAN FIELD THEORY

We will leave aside the quantum fields for while, and consider classical field theory in its Lagrangian formulation.

5.1 The Action Principle

A classical field theory may be defined in terms of an action S that is given as an integral over space and time of a Lagrangian density $\mathcal{L}$. This Lagrangian density $\mathcal{L}$ is a Lorentz invariant function of the field ϕ and its derivative $\partial_\mu \phi$. Thus in general the action S takes the form

$$S = \int d^4x \, \mathcal{L}(\phi(x), \partial_\mu \phi(x)). \tag{48}$$

According to the *action principle* one obtains the classical field equations by requiring that the action is stationary, that means it does not change under infinitesimal variations of the fields $\phi \to \phi + \delta\phi$. For general values of the field ϕ the the action S changes under such a variation by a small amount δS given by

$$\begin{aligned} \delta S = & \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) \right) \\ & \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta\phi \end{aligned} \tag{49}$$

where we dropped a surface terms after the partial integration. In this way we find that the variation of the action vanishes provided the field $\phi(x)$ satisfies the Euler-Lagrange equation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{50}$$

Note that to obtain a linear Euler-Lagrange equation the Lagrangian must be quadratic in the fields ϕ and its derivatives. But in general one can write down Lagrangians that , besides a quadratic terms, involve higher order powers. These Lagrangians will give rise to non-linear field equations. It is in this way that interactions are introduced in the Lagrangian formalism.

We have seen that the fields ϕ and ψ associated with particles with spin zero and spin 1/2 satisfy the Klein-Gordon and Dirac equation. These wave equations can in fact be derived from an action principle. Because the wave equations are linear it is not difficult to construct the corresponding Lagrangrians $\mathcal{L}$. For the Klein Gordon field the Lagrangian is

$$\mathcal{L}_\phi = \tfrac{1}{2} \partial_\mu \phi \partial^\mu \phi - \tfrac{1}{2} m^2 \phi^2, \tag{51}$$

One easily checks that $\frac{\delta \mathcal{L}}{\delta \phi} = m^2 \phi$ and $\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} = -\partial_\mu \phi$ and thus the Euler-Lagrange equation indeed reproduces the Klein-Gordon equation. In a similar way one can find the Lagrangian for the Dirac field

$$\mathcal{L}_\psi = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{52}$$

where $\overline{\psi} = \psi^\dagger \gamma_0$.

5.2 Noether's Theorem: An Example

Charged particles with spin, such as the $\pi^{\pm}$ pions, are described by a complex scalar field φ. For this field the Lagrangian is

$$\mathcal{L}_\varphi = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi, \tag{53}$$

Notice that the Lagrangian is real, despite of the fact that φ is complex. Moreover, when we multiply the field φ by a complex phase factor

$$\varphi \to e^{i\chi}\varphi \tag{54}$$

the Lagrangian $\mathcal{L}_\varphi$ will remain the same because $(e^{i\chi})^* e^{i\chi} = 1$. This has an important consequence. Namely, suppose that χ is a function of the space and time coordinate x^μ. In that case the transformation (54) is not a symmetry of the Lagrangian or the action. Yet, when the fields satisfy the equation of motion, we know that the action should be invariant under infinitesimal transformations $\varphi \to \varphi + \delta\varphi$. For an infinitesimal phase rotation (54) with a small parameter $\delta\chi$ the field φ varies with $\delta\varphi = i\delta\chi\varphi$. After inserting this variation in the action one can easily see that all terms without derivatives on $\delta\chi$ cancel. The terms that are left over are

$$\delta S = ie \int d^4x \partial^\mu (\delta\chi)\,(\varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^*)$$

From the action principle we know that this variation should vanish, and therefore we conclude that the expression

$$j^\mu_e = ie(\varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^*) \tag{55}$$

must satisfy the conservation law

$$\partial_\mu j^\mu_e = 0. \tag{56}$$

This can also be checked directly from the equation of motion. The current $j^e_\mu = (\vec{j}^e, \rho_e)$ represents the electric current and charge density. The fact that the presence of a conserved current is associated with a symmetry of the Lagrangian is known as Noether's theorem. For a complex Dirac field, which describes the electron and positron, one can in a similar way determine the expression for the electric current density. It is

$$j^\mu_e = e\overline{\psi}\gamma^\mu\psi \tag{57}$$

This current will play an important role in describing the interactions with the electromagnetic field.

5.3 Photons and the Electro Magnetic Field

The photon is a particle with spin one, and is described by a vector field $A_\mu = (\vec{A}, V)$. These fields $\vec{A}$ and V are in fact the vector and scalar potential for the electric field $\vec{E} = \partial_t \vec{A} - \vec{\nabla} V$ and magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$. Already in the previous century Maxwell wrote down the field equations for $\vec{E}$ and $\vec{B}$

$$\begin{aligned} \vec{\nabla} \times \vec{B} - \partial_t \vec{E} &= \vec{j}_e \\ \vec{\nabla} \cdot \vec{E} &= \rho_e \end{aligned} \tag{58}$$

where ρ_e and $\vec{j}_e$ are the electric charge and current densities. The fact that these equations are Lorentz invariant was discovered only later, and led to the development of special relativity. Nowadays, we write often the Maxwell equations directly in a covariant form in terms of the electromagnetic field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{59}$$

which combines the electric and magnetic field: $F_{0i} = E_i$, $F_{ij} = \epsilon_{ijk} B_k$. The equation (58) can be combined as

$$\partial^\mu F_{\mu\nu} = j^e_\nu \tag{60}$$

Notice that this equation only has solutions when the current j^e_ν satisfies the conservation law (56), because for the left-hand-side of this equation we automically have $\partial^\nu \partial^\mu F_{\mu\nu} = 0$. The Maxwell equations can again be derived from an action principle. The Lagrangian is given as a function of $A_\mu(x)$ by

$$\mathcal{L}_A = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + A_\mu j^\mu_e \tag{61}$$

It is not difficult to verify that the Euler-Lagrange equations give the equation (60). For example, the linear term in A_μ directly leads to the right hand side in terms of the electric current.

6. QUANTUM ELECTRO DYNAMICS

Quantum Electro Dynamics (QED) describes the electro-magnetic interactions between electrons and positrons. By combining the results of the previous section we can easily construct an action for the electron and positron field together with the electromagnetic field. Namely, we simply take the sum of the Lagrangians(52) and (61)

$$\mathcal{L}_{QED} = \mathcal{L}_A + \mathcal{L}_\psi$$

and identify the electric current with the Noether current (57), that is $j^\mu_e = e\overline{\psi}\gamma^\mu\psi$. Combining these ingredients we can write the Lagrangian of quantum electro dynamics as

$$\mathcal{L}_{QED} = -\frac{1}{4}F^2_{\mu\nu} - \overline{\psi}\,(i\gamma^\mu D_\mu - m)\,\overline{\psi} \tag{62}$$

where

$$D_\mu = (\partial_\mu - ieA_\mu). \tag{63}$$

is called the covariant derivative.

6.1 Gauge Invariance of the Lagrangian

A very important property of this Lagrangian is that it is invariant under local *gauge transformations*. These tranformations act on the field A_μ as

$$A_\mu \to A_\mu + \tfrac{1}{e}\partial_\mu \chi. \tag{64}$$

and simultaneously on the fermion field ψ and $\overline{\psi}$ as complex phase rotations

$$\begin{aligned} \psi &\to e^{i\chi}\psi, \\ \overline{\psi} &\to \overline{\psi} e^{-i\chi} \end{aligned} \tag{65}$$

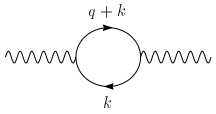
First, one easily checks that the field strength is invariant: $F_{\mu\nu} \to F_{\mu\nu} + \frac{1}{e}(\partial_\mu\partial_\nu\chi - \partial_\nu\partial_\mu\chi) = F_{\mu\nu}$. Next, the fact that the derivative ∂_μ on the fermion field is replaced by the 'covariant derivative' D_μ guarantees the invariance of the kinetic term of the fermions because it follows from (63) and (64) that under a gauge transformation $D_\mu \to e^{i\chi} D_\mu e^{-i\chi}$ and so

$$D_\mu \psi \to e^{i\chi} D_\mu \psi \tag{66}$$

This phase factor cancels against the one of $\overline{\psi}$. The significance of the gauge invariance is that it ensures that the photon only has two physical polarizations and remains massless.

6.2 The Feynman Rules of QED

Given the form of the Lagrangian one can read of the Feynman rules in a simple manner by writing the action in terms of Fourier modes. The quadratic terms in the fields describe the kinetic terms and determines the form of the propagator $\Delta(p)$. To be precise, the propagator is given by the inverse of the kinetic operator written in momentum space. The propagators for the electron-positron field is



$$i\frac{\gamma^\mu p_\mu + m}{p^2 - m^2 + i\epsilon}$$

which was already derived in section 2.4. To find the propagator for the photon is somewhat more tricky due to the gauge invariance. Because of this the kinetic term would not give an invertible expression in momentum space. The right procedure is to add a so-called gauge-fixing term

$$\mathcal{L}_{g.f.} = -\tfrac{1}{2}\xi^{-1}(\partial^\mu A_\mu)^2 \tag{67}$$

So that the kinetic term is invertible. This gives the following photon propagator

$$\Delta_{\mu\nu}(q) = -i\frac{\eta_{\mu\nu} + (1-\xi)q_\mu q_\nu/q^2}{q^2 + i\epsilon} \tag{68}$$

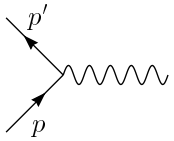
Gauge invariance ensures that the result is independent of ξ. We can use this fact to choose $\xi = 1$, so that the expression simplifies. We will use as photon propagator

$$\frac{-i\eta_{\mu\nu}}{q^2 + i\epsilon}$$

The Lagrangian (62) contains only one term that is not quadratic in the fields. This interaction term is

$$\mathcal{L}_{int} = eA_\mu \overline{\psi}\gamma^\mu\psi \tag{69}$$

We conclude that QED has only one vertex, namely one with two fermion lines and one photon line. The corresponding factor is



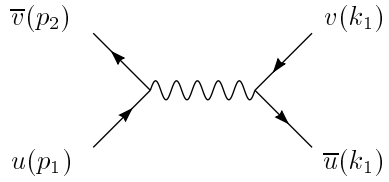
$$ie\gamma^\mu$$

The external lines for the electrons are represesented by their wave functions $u(\vec{p}$ or $v(\vec{p})$ for incoming lines, and by $\overline{u}(p)$ and $\overline{v}(\vec{p}$ for outgoing lines. Similarly, the external photons

have a factor $\epsilon_\mu(k)$ that describes their polarization. The physical polarization satisfy $k^\mu \epsilon_\mu = 0$. Furthermore, due to the gauge invariance the Feynman amplitudes must be invariant under $\epsilon_\mu \sim \epsilon_\mu + \lambda k_\mu$. This implies that longitudinal photons with $\epsilon_\mu = \lambda k_\mu$ decouple, and hence the photon has only two physical polarizations. Finally, each loop momentum integral has a normalization factor $(2\pi)^{-4}$.

6.3 A Tree Level Calculation

As an illustration of the Feynman rules let us consider the scattering amplitude for the process $e^+e^- \to \mu^+\mu^-$. At tree level there is only one Feynman diagram that contributes. It is given by



Feynman diagram for $e^+e^- \to \mu^+\mu^-$

In words what happens is that the electron and positron annihilate and produce a virtual photon. This photon propagates for a while and then decays in to the muon-pair μ^+ and μ^-. The amplitude for this process is given by

$$\mathcal{A}^{ab}_{a'b'}(\vec{p}_1,\vec{p}_2;\vec{k}_1,\vec{k}_2) = ie\,\overline{v}^b(\vec{p}_1)\gamma^\mu u^a(\vec{p}_1)\;\frac{-i\eta_{\mu\nu}}{(p_1+p_2)^2}\;ie\,\overline{u}^{a'}(\vec{k}_1)\gamma^\nu v^{b'}(\vec{k}_2) \tag{70}$$

The transition rate W is obtained by squaring this amplitude. For unpolarized electrons and positrons one should average over the initial polarizations a and b. Also, when one ignores the muon-polarization one has to sum over a' and b'. Combining these ingredients one finds that the transition rate take the form

$$dW = \frac{1}{4}\sum_{a,b,a',b'} |\mathcal{A}^{ab}_{a'b'}(\vec{p}_i;\vec{k}_j)|^2 = \frac{4e^4}{s^2} L^{\mu\nu}(p_1,p_2)L_{\mu\nu}(k_1,k_2) \tag{71}$$

where

$$L_{\mu\nu}(p_1,p_2) = -\frac{1}{4}\sum_{a,b} \overline{u}_a(\vec{p}_1)\gamma_\mu v_b(\vec{p}_2)\overline{v}^*_b(\vec{p}_2)\gamma_\nu u^*_a(\vec{p}_1) \tag{72}$$

Here we also introduced the Mandelstam variable $s = (p_1+p_2)^2$ which is equal to the square of the center of mass energy. Now, using identities like (33) and after evaluating the trace of the gamma matrices one finds

$$L^{\mu\nu}(p_1,p_2) = p_1^\mu p_2^\nu + p_1^\nu p_2^\mu + \eta^{\mu\nu}(p_1\cdot p_2 + m^2) \tag{73}$$

We will now assume that the energies are large so that we ignore the terms with the mass m^2. Inserting the expression for $L^{\mu\nu}$ in to

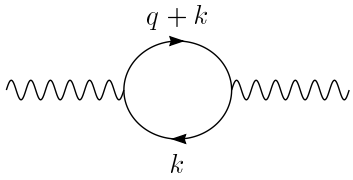
$$dW = 2e^4\frac{t^2+u^2}{s^2} \tag{74}$$

where $t = (p_1-k_1)^2$ and $u = (p_1-k_2)^2$ are the two other Mandelstam variables. This result can now be compared to experiments, and gives a way of determining the value of the coupling constant e.

6.4 Loop Corrections

At one loop there are four kinds of Feynman diagrams that give corrections to the tree level result. First there is the possibility that the electron emits a photon which it again re-absorbs before it annihilates with the positron. This gives the electron self-energy. Then there is a so-called vertex correction where a photon is emitted by the electron and absorbed by the positron before both annihilate. Another possibility is that the emitted photon is absorbed only later by one of the produced muons.

The remain type of correction, which we will consider in much more detail, is the so-called vacuum polarization graph. In words it describes a process where the photon that is created out to the electron and positron produces first a virtual lepton (e^+ e^- or μ^+ μ^-) pair. This virtual pair again annihilates and produces a photon which then in second instance produce the muon pair. So this correction comes from an additional fermion loop inside the photon propagator.



Vacuum polarization diagram

This diagram will change the value of the electric charge e and in fact make it dependent on the center of mass energy of the initial electron and positron. Intuitively what happens is that the vacuum contains fluctuations of lepton pairs ℓ^- and ℓ^+. Through the electric force the positively charged virtual leptons ℓ^+ move somewhat towards the electron while the virtual ℓ^- particles move towards the positron. In this way the virtual fluctuations "polarize" the vacuum and screen the electric charge e of both the electron and the positron. This effect becomes more important when the center of mass energy of the colliding electron and positron is large. This may be understood intuitively from the fact that to form a virtual electron-positron or muon pair the vacuum has to "borrow" part of this center of mass energy and so the number of virtual pairs increases when more energy is available[3)].

The vacuum polarization graph only affects the part of the diagram that represents the photon propagator. In fact, without much additional work one can include also higher loop corrections that correspond to repeated vacuum polarization diagrams. These can be summed up by a geometric series. Its effect can be included by shifting the inverse propagator $\Delta^{-1}_{\mu\nu}(q)$ by a term as follows

$$\Delta^{-1}_{\mu\nu}(q) = iq^2\eta_{\mu\nu} \to iq^2\left(\eta_{\mu\nu} + \Pi_{\mu\nu}(q)\right) \tag{75}$$

where we again put $\xi = 1$ and

$$q^2\Pi_{\mu\nu}(q^2) = \frac{-e^2}{(2\pi)^4}\int d^4k\,\frac{\mathrm{Tr}\left(\gamma_\mu(\gamma_\lambda k^\lambda + m)\gamma_\nu(\gamma_\sigma(q+k)^\sigma + m)\right)}{(k^2-m^2)((q+k)^2-m^2)} \tag{76}$$

3) Mysteriously enough, this intuition fails for non-abelian gauge fields, where one finds an "anti-screening" effect that is responsible for asymptotic freedom.

It follows from Lorentz invariance and gauge invariance (decoupling of the longitudinal polarization of the photon) that this expression must be of the form

$$\Pi_{\mu\nu}(q^2) = (q^2\eta_{\mu\nu} - q^\mu q^\nu)\Pi(q^2) \tag{77}$$

This can be verified explicitely by evaluating the gamma trace and doing some manipulations in the integral. Fruther, we will again assume that the energy carried by the photon is much larger that the electron mass, and so we drop the terms depending on m. In this way one finds

$$\Pi(q^2) = \frac{-e^2}{(2\pi)^4}\int d^4k\,\frac{1}{k^2(q+k)^2} \tag{78}$$

This integral is infinite. This is of course rather disturbing result, but fortunately there is still a way to extract a sensible answer. First one has to *regulate* the integral so that it is finite, and then apply a so-called *renormalization* procedure. Here we will not go in too much detail, but we just want make clear what the essential idea behind renormalization.

6.5 Regularization and Renormalization

There are various way to regulate the above integral expression. The simplest fashion is to analytically continue to euclidean space and to introduce a momentum cut off Λ by restricting the integral over k to values with $|k| < \Lambda$. Again omitting some details, one gets

$$\begin{aligned}\Pi(q^2) &= -\frac{e^2}{12\pi^2}\int_{q^2}^{\Lambda^2}\frac{d(k^2)}{k^2} \\ &= \frac{e^2}{12\pi^2}\ln\left(\frac{q^2}{\Lambda^2}\right)\end{aligned} \tag{79}$$

This one-loop correction can now be inserted in the Feynman graph for the full process, and in this way added to the tree-level result. Because it only corrects the photon propagator the result is of the same form as the tree-level, but with an additional s dependent factor. This eventually leads to

$$dW = 2e^4\frac{t^2+u^2}{s^2}\left(1 - \frac{e^2}{12\pi^2}\ln\left(\frac{s}{\Lambda^2}\right)\right)^{-2} \tag{80}$$

Now there appears to be a problem with this expression. Namely, it depends on the cut-off Λ, which was chosen arbitrarily. In particular, when we would send the cut-off $\Lambda \to \infty$ we would again find a divergent answer, at least when we keep e fixed. The solution to this problem is that the coupling constant e that appears in this expression should not really be identified with the physical value of the electric charge that we measure in the laboratory. Therefore, we are allowed to make this 'bare' coupling $e = e_0(\Lambda)$ dependent on the cut-off Λ in such a way that the physical observables like dW are independent of Λ.

Suppose we first do the experiment at a certain center of mass energy $s = \mu^2$. Then at this value of s we can now use the tree-level expression (74) to define the *physical* coupling constant $e(\mu)$ by

$$dW_{|s=\mu^2} = 2e^4(\mu)\frac{t^2+u^2}{s^2} \tag{81}$$

Comparing this to the one loop result (80) one concludes that

$$e^2(\mu) = e_0^2(\Lambda)\left(1 - \frac{e_0^2(\Lambda)}{12\pi^2}\ln\left(\frac{\mu^2}{\Lambda^2}\right)\right)^{-1} \tag{82}$$

where we made explicit that the bare coupling $e_0^2(\Lambda)$ depends on the cut-off. This equation now tells us what $e_0^2(\Lambda)$ is in terms of the physical coupling $e(\mu)$. Next we can re-express the result for the physical observable dW at other values for s in terms of the physical coupling $e(\mu)$. In this way one finds that cut-off Λ disappears from the expression for dW, because it follows from (82) that

$$e_0^2(\Lambda)\left(1 - \frac{e_0^2(\Lambda)}{12\pi^2}\ln\left(\frac{s}{\Lambda^2}\right)\right)^{-1} = e^2(\mu)\left(1 - \frac{e^2(\mu)}{12\pi^2}\ln\left(\frac{s}{\mu^2}\right)\right)^{-1} \tag{83}$$

and so we can write the transition probability in terms of the physical coupling constant as

$$dW = 2e^4(\mu)\frac{t^2+u^2}{s^2}\left(1 - \frac{e^2(\mu)}{12\pi^2}\ln\left(\frac{s}{\mu^2}\right)\right)^{-2} \tag{84}$$

This expression is identical to (80) but now the cut-off Λ is replaced by the mass scale μ, which is called the renormalization scale. Still there appears to be the problem that the answer for dW depends on the arbitrary scale μ, while we know that the physical value of dW must be independent of μ. However, this condition precisely determines the way that $e(\mu)$ depends on μ. So finally, we conclude that the effect of the one-loop vacuum polarization can be summarized by equation (81) with a *running coupling constant* $e(\mu)$. Given the value of $e(\mu)$ at low energies, say at $\mu_0 \sim$ a few eV, we can determine its value at other values from the condition that value of dW remains unchanged. This gives

$$e^2(\mu) = e^2(\mu_0)\left(1 - \frac{e^2(\mu_0)}{12\pi^2}\ln\left(\frac{\mu^2}{\mu_0^2}\right)\right)^{-1} \tag{85}$$

At low energies we know the value of the coupling constant very well: we have $\alpha(\mu_0) = e^2(\mu_0)/4\pi = 1/137$. The value at higher energies follows from the above relation which may be rewritten in terms of the fine structure constant $\alpha(\mu) = e^2(\mu)/4\pi$ as

$$\frac{1}{\alpha(\mu)} = \frac{1}{\alpha(\mu_0)} - \frac{1}{3\pi}\ln(\frac{\mu^2}{\mu_0^2}). \tag{86}$$

In fact, at high energies one also has to take in to account the contribution to the vacuum polarization of other charged particles, in particular the muon. This leads to additional contributions that change the coefficient $\frac{1}{3\pi}$ somewhat. This running of the fine structure constant has been confirmed by high energy experiments, in particular at LEP.